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A Finite Variable Difference Relaxation Scheme for hyperbolic–parabolic equations

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ABSTRACT

Using the framework of a new *relaxation system*, which converts a nonlinear viscous conservation law into a system of linear convection–diffusion equations with nonlinear source terms, a finite variable difference method is developed for nonlinear hyperbolic–parabolic equations. The basic idea is to formulate a finite volume method with an optimum spatial difference, using the Locally Exact Numerical Scheme (LENS), leading to a Finite Variable Difference Method as introduced by Sakai [Katsuhiro Sakai, A new finite variable difference method with application to locally exact numerical scheme, Journal of Computational Physics, 124 (1996) pp. 301–308.], for the linear convection–diffusion equations obtained by using a relaxation system. Source terms are treated with the well-balanced scheme of Jin [Shi Jin, A steady-state capturing method for hyperbolic systems with geometrical source terms, Mathematical Modeling Numerical Analysis, 35 (4) (2001) pp. 631–645]. Bench-mark test problems for scalar and vector conservation laws in one and two dimensions are solved using this new algorithm and the results demonstrate the efficiency of the scheme in capturing the flow features accurately.

1. Introduction

Numerical solution of hyperbolic and parabolic partial differential equations has reached a state of maturity in the past few decades. A major contribution to this progress came from the researchers in Computational Fluid Dynamics (CFD). The numerical solution of Euler equations of gas dynamics and the shallow water equations, which represent hyperbolic vector conservation laws, is now considered to be an established field, with several innovative numerical methods having been introduced in the past few decades. Some reviews of this history are available in [16,17,25,47,27]. One major focus in this development has been the introduction of higher order accurate methods which are stable and are free of numerical oscillations. The *Total Variation Diminishing* (TVD) schemes with limiters are especially designed for this purpose. However, difficulties still remain with this approach: getting uniformly higher order accuracy in all parts of the computational domain without clipping of the extrema, especially in multi-dimensions and with unstructured meshes, is hard to achieve and the research is still continuing in this area, as can be seen from the large number of papers continuously being published. The reader is referred to the books edited by Hussaini, van Leer and Rosendale [19], Barth and Deconink [5] and the references therein for a glimpse of these developments.

One of the essential difficulties associated with higher order schemes for convection dominated flow simulations is the nonmonotonicity of the solutions, manifesting as oscillations or wiggles, especially near high gradient regions or discontinuities. In this context, an important early development was the Godunov theorem [10] in which it was shown that linear

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higher order schemes would necessarily lead to nonmonotone solutions. A way to circumvent this limitation is to make the coefficients nonlinear, by making the coefficients depend on the solution variables. The Flux Corrected Transport (FCT) algorithm of Boris and Book [6–8,53] and the Monotone Upstream Schemes for Conservation Laws (MUSCL) methodology of van Leer [48–51] are based on this strategy (see also [13]). Harten introduced the popular Total Variation Diminishing (TVD) schemes [14] by providing a rigorous mathematical foundation for this approach which was further developed by Sweby [46] and a lot of other researchers (see [19] for details). Goodman and Leveque [11] showed that multi-dimensional TVD schemes are no better than first order accurate schemes. Later, Essentially NonOscillatory (ENO) schemes were developed to overcome some of the deficiencies of the TVD schemes like clipping the extrema [15]. A detailed account of the further development of ENO schemes is given by Shu [44]. In spite of their advantages over TVD methods, ENO methods are not so convenient to extend to multi-dimensions. The development of higher order schemes is far from complete and the research in this area is still continuing; according to Roe [39], rigor is not yet to be found in all aspects of higher order schemes. In this context, it is worth looking for alternative approaches.

In the process of discretization of the convection dominated equations in CFD, it is considered advantageous to mimic the properties of the exact solutions of the original equations (when they are available) so that the resulting discretized equations have better chance to converge to the physically relevant solutions. For example, the upwind methods mimic the exact solutions of convection equations as closely as possible in a finite difference framework. An interesting alternative to this strategy is to develop a numerical method in which the coefficients in the difference equation satisfy the exact solution of the original convection (or convection–diffusion) equation. Some numerical methods using this strategy were developed by Allen and Southwell [1], Günther [12] and Sakai [40]. A related idea is used in *Nonstandard Finite Difference Methods* of Mickens [31] in which the difference equations have the same general solutions as the associated differential equations. An attempt to apply this approach to nonlinear convection equations and linear systems of convection equations is given in [52]. These methods produce results which are very close to the exact solutions, many times producing much superior results compared to traditional finite difference methods.

The Finite Variable Difference Method (FVDM) of Sakai [41] is one such interesting alternative to the traditional approaches. In this method, a nonoscillatory algorithm is developed for convection–diffusion equations, based on a variable mesh increment, with the optimal spatial difference being determined by minimizing the variance of the solution by choosing the roots of the difference equation to be nonnegative. Sakai [41,43] has demonstrated the efficiency of the FVDM for linear convection–diffusion problems and derived some new schemes based on this strategy for the 1-D Burgers equation, in which the hyperbolic terms are nonlinear. An important feature of the FVDM is that a nonoscillatory scheme is formulated explicitly based on the exact solution of convection–diffusion equations, a feature not shared by the convectional higher order schemes such as TVD methods. Note that in this scheme, the formulation of a nonoscillatory scheme is done directly for convection–diffusion equations, whereas most of the TVD methods are formulated only for convection equations. Another important feature of the FVDM is that the drive towards accuracy in developing a nonoscillatory scheme is based on minimizing the variance of the solution, variance being the deviation from the exact solution, which is more reasonable to use compared to the conventional derivation based on Taylor series expansions in the finite difference methods [41]. Yet another interesting feature of the FVDM is that formulation of the scheme is not based explicitly on artificial viscosity. Sakai [41] has demonstrated that the oscillations in the solutions are completely avoided by the FVDM in the case of steady equations and only very mild oscillations appear in the unsteady case. Because of all the above features, the FVDM is selected in this work, as an interesting alternative for study in developing accurate numerical methods for hyperbolic–parabolic partial differential equations representing conservation/balance laws.

Extending the FVDM directly to the unsteady and nonlinear Burgers equation seems to be nontrivial. So far, the FVDM has also not been applied to the hyperbolic systems of conservation laws. In this study, we extend Sakai's Finite Variable Difference Method to nonlinear Burgers equation, and also to a hyperbolic system of conservation laws (shallow water equations), by coupling this method to a new *relaxation system* which modifies the relaxation system of Jin and Xin [20], while linearizing the nonlinear hyperbolic–parabolic equations. We utilize the strategy used by Sakai, by applying FVDM to the Locally Exact Numerical Scheme in which the difference coefficients are determined such that the resulting difference equation satisfies the exact solution of the convection–diffusion equation, in our frame work of a novel relaxation system applied to the nonlinear convection–diffusion equations. This *Finite Variable Difference Relaxation Scheme* (FVDRS) is tested on some benchmark test problems for 1-d inviscid Burgers equation, 1-D viscous Burgers equation, 2-D inviscid Burgers equation, 2-D viscous Burgers equation and shallow water equations in both one and two dimensions. The results demonstrate the efficiency of this *Finite Variable Difference Relaxation Method*. It is worth noting that our method is not based on Riemann solvers, which are reported to be associated with a list of failures [34].

2. A relaxation system for viscous Burgers equation

Jin and Xin [20] introduced a relaxation system for hyperbolic equations like inviscid Burgers equation or Euler equations. A relaxation system provides a vanishing viscosity model for nonlinear hyperbolic conservation laws by replacing the nonlinear hyperbolic (convection) terms with linear hyperbolic terms with a stiff nonlinear source term which represents a mathematical relaxation process. The relaxation schemes, based on a relaxation system, are interesting alternatives to traditional schemes for solving hyperbolic conservation laws. The reader is referred to [20,32,2,26,36,37,4] for some numerical methods based on relaxation systems. Extending the relaxation systems to hyperbolic–parabolic systems representing convection–diffusion systems or viscous conservation laws is a nontrivial task. Lions and Toscani [28], Jin, Pareschi and Toscani [21], Jin [22], Raghurama Rao [35] (see also [38]) and Aregba-Driollet, Natalini and Tang [3] introduced some relaxation systems for hyperbolic–parabolic equations and corresponding numerical methods for viscous conservation laws. In this section, we introduce a new relaxation system for the viscous Burgers equation, by treating the viscous term as a source term.

Consider the viscous Burgers equation, given by

$$\frac{\partial u}{\partial t}+\frac{\partial g(u)}{\partial x}=\frac{\partial g_v(u)}{\partial x} \tag{1}$$

Here the flux $g(u)$ is a nonlinear function of the dependent variable u (since $g(u)=\frac{1}{2}u^2$) and $g_v(u)$ is viscous flux ($g_v(u)=\nu\frac{\partial u}{\partial x}$). The above nonlinear partial differential equation is converted into two linear partial differential equations with a nonlinear relaxation term by introducing a new variable v (for the nonlinear flux $g(u)$) as

$$\frac{\partial u}{\partial t}+\frac{\partial v}{\partial x}=\frac{\partial g_v(u)}{\partial x} \tag{2}$$

$$\frac{\partial v}{\partial t}+\lambda^2\frac{\partial u}{\partial x}=-\frac{[v-g(u)]}{\epsilon} \tag{3}$$

where λ is a positive constant (relaxation parameter) and ϵ is a small parameter such that $\epsilon\to 0$. When $\epsilon\to 0$, the second equation of the relaxation system (3) gives $v=g(u)$, which when substituted into the first Eq. (2), gives back the viscous Burgers equation (1). Thus, solving (2) and (3) in the limit of $\epsilon\to 0$ is equivalent to solving (1). The advantage lies in the fact that the relaxation system (2) and (3) is linear in convection terms (on the left hand side) and is easy to deal with. The right hand side of the relaxation system is still nonlinear due to the relaxation term, but if we use the splitting method, the above relaxation system can be split into a linear system of convection equations and an ordinary differential equation (containing the nonlinear term), both of which can be easily solved.

The above relaxation system can be written in vector form as

$$\frac{\partial Q}{\partial t}+A\frac{\partial Q}{\partial x}=H \tag{4}$$

where

$$Q=\begin{bmatrix}u\\ v\end{bmatrix},\quad A=\begin{bmatrix}0 & 1\\ \lambda^2 & 0\end{bmatrix}\quad\text{and}\quad H=\begin{bmatrix}\frac{\partial g_v(u)}{\partial x}\\ \frac{-[v-g(u)]}{\epsilon}\end{bmatrix} \tag{5}$$

Since Eq. (4) has linear hyperbolic terms on the left hand side, we can use the characteristic variables to obtain a set of decoupled equations. We can write

$$A=R\Lambda R^{-1}\quad\text{and thus}\quad \Lambda=R^{-1}AR \tag{6}$$

where R is the matrix of right eigenvectors of A, R^{-1} is its inverse and Λ is a diagonal matrix with eigenvalues of A as its elements. The expressions for R, R^{-1} and Λ are given by

$$R=\begin{bmatrix}1 & 1\\ -\lambda & \lambda\end{bmatrix},\quad R^{-1}=\begin{bmatrix}\frac{1}{2} & \frac{-1}{2\lambda}\\ \frac{1}{2} & \frac{1}{2\lambda}\end{bmatrix}\quad\text{and}\quad \Lambda=\begin{bmatrix}-\lambda & 0\\ 0 & \lambda\end{bmatrix} \tag{7}$$

Introducing f as a characteristic variable vector given by

$$f=R^{-1}Q \tag{8}$$

we obtain from the vector form of the relaxation system (4) the decoupled system as

$$\frac{\partial f}{\partial t}+\Lambda\frac{\partial f}{\partial x}=R^{-1}H \tag{9}$$

where

$$f=\begin{bmatrix}f_1\\ f_2\end{bmatrix}=\begin{bmatrix}\frac{u}{2}-\frac{v}{2\lambda}\\ \frac{u}{2}+\frac{v}{2\lambda}\end{bmatrix}\quad\text{and}\quad R^{-1}H=\begin{bmatrix}\frac{1}{2}\frac{\partial g_v(u)}{\partial x}+\frac{v-g(u)}{2\epsilon}\\ \frac{1}{2}\frac{\partial g_v(u)}{\partial x}-\frac{v-g(u)}{2\epsilon}\end{bmatrix} \tag{10}$$

Thus we obtain two decoupled equations as

$$\begin{aligned}\frac{\partial f_1}{\partial t}-\lambda\frac{\partial f_1}{\partial x}&=\frac{1}{2}\frac{\partial g_v(u)}{\partial x}+\frac{[v-g(u)]}{2\lambda\epsilon}\\ \frac{\partial f_2}{\partial t}+\lambda\frac{\partial f_2}{\partial x}&=\frac{1}{2}\frac{\partial g_v(u)}{\partial x}-\frac{[v-g(u)]}{2\lambda\epsilon}\end{aligned} \tag{11}$$

The original variable u and v can be recovered as

$$u = f_1 + f_2 \quad \text{and} \quad v = \lambda(f_2 - f_1) \tag{12}$$

and thus we obtain

$$\frac{\partial g_v(u)}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} = \nu \frac{\partial^2 f_1}{\partial x^2} + \nu \frac{\partial^2 f_2}{\partial x^2} \tag{13}$$

Thus, the decoupled equations become

$$\frac{\partial f_1}{\partial t} - \lambda \frac{\partial f_1}{\partial x} = \frac{1}{2}\nu \frac{\partial^2 f_1}{\partial x^2} + \frac{1}{2}\nu \frac{\partial^2 f_2}{\partial x^2} + \frac{[v - g(u)]}{2\lambda\epsilon} \tag{14}$$

$$\frac{\partial f_2}{\partial t} + \lambda \frac{\partial f_2}{\partial x} = \frac{1}{2}\nu \frac{\partial^2 f_2}{\partial x^2} + \frac{1}{2}\nu \frac{\partial^2 f_1}{\partial x^2} - \frac{[v - g(u)]}{2\lambda\epsilon} \tag{15}$$

Eqs. (14) and (15) are convection diffusion equations with source terms. Note that the convection parts of the equations are now linear, unlike the original viscous Burgers equation in which the convection terms are nonlinear. Let us now rewrite these equations by introducing new variables F_1 and F_2 as

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}u - \frac{1}{2\lambda}g(u) \\ \frac{1}{2}u + \frac{1}{2\lambda}g(u) \end{bmatrix} \tag{16}$$

Thus, we obtain

$$\frac{\partial f_1}{\partial t} - \lambda \frac{\partial f_1}{\partial x} = \frac{1}{2}\nu \frac{\partial^2 f_1}{\partial x^2} + \frac{1}{2}\nu \frac{\partial^2 f_2}{\partial x^2} - \frac{[f_1 - F_1]}{\epsilon} \tag{17}$$

$$\frac{\partial f_2}{\partial t} + \lambda \frac{\partial f_2}{\partial x} = \frac{1}{2}\nu \frac{\partial^2 f_2}{\partial x^2} + \frac{1}{2}\nu \frac{\partial^2 f_1}{\partial x^2} - \frac{[f_2 - F_2]}{\epsilon} \tag{18}$$

Note the similarity of the above equations to the discrete velocity Boltzmann equation, with F representing a Maxwellian distribution.

3. Chapman–Enskog type expansion for the relaxation system

In this section, the stability of the new relaxation system introduced in the previous section is studied by using Chapman–Enskog type expansion. The viscous Burgers equation is

$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = \frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x}\right) \tag{19}$$

The relaxation system for the above nonlinear convection–diffusion equation is given by the following two equations:

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x}\right) \tag{20}$$

$$\frac{\partial v}{\partial t} + \lambda^2 \frac{\partial u}{\partial x} = -\frac{1}{\epsilon}[v - g(u)] \tag{21}$$

From (21), we can obtain

$$v = g(u) - \epsilon\left[\frac{\partial v}{\partial t} + \lambda^2 \frac{\partial u}{\partial x}\right] \tag{22}$$

which can be written as

$$v = g(u) + O(\epsilon) \tag{23}$$

Therefore

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}[g(u) + O(\epsilon)] = \frac{\partial g(u)}{\partial u}\frac{\partial u}{\partial t} + O(\epsilon) = \frac{\partial g(u)}{\partial u}\left[-\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x}\right)\right] + O(\epsilon) \text{ (from (20))} \\ &= \frac{\partial g(u)}{\partial u}\left[-\frac{\partial}{\partial x}\{g(u) + O(\epsilon)\} + \frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x}\right)\right] + O(\epsilon) = \frac{\partial g(u)}{\partial u}\left[-\frac{\partial g(u)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x}\right)\right] + O(\epsilon) \end{aligned} \tag{24}$$

or

$$\frac{\partial v}{\partial t} = -\{a(u)\}^2 \frac{\partial u}{\partial x} + \{a(u)\}\left\{\frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x}\right)\right\} + O(\epsilon) \tag{25}$$

where $a(u)$ is the wave speed of the convection–diffusion Eq. (19), given by

$$a(u) = \frac{\partial g(u)}{\partial u} \tag{26}$$

Substituting (25) in (22), we obtain

$$v = g(u) - \epsilon\left[-\{a(u)\}^2\frac{\partial u}{\partial x} + \{a(u)\}\left\{\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right)\right\} + O(\epsilon) + \lambda^2\frac{\partial u}{\partial x}\right]$$
$$= g(u) - \epsilon\left[\frac{\partial u}{\partial x}\left\{\lambda^2 - \{a(u)\}^2\right\} + \{a(u)\}\left\{\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right)\right\}\right] + O(\epsilon^2) \tag{27}$$

Therefore

$$\frac{\partial v}{\partial x} = \frac{\partial g(u)}{\partial x} - \epsilon\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\left\{\lambda^2 - \{a(u)\}^2\right\} + \{a(u)\}\left\{\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right)\right\}\right] + O(\epsilon^2) \tag{28}$$

Substituting (28) in (20), we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right) + \epsilon\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\left\{\lambda^2 - \{a(u)\}^2\right\}\right] + \epsilon\frac{\partial}{\partial x}\left[\{a(u)\}\left\{\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right)\right\}\right] + O(\epsilon^2) \tag{29}$$

The above expression (29), obtained from Chapman–Enskog type expansion for the relaxation system, represents the vanishing dissipation–dispersion model for the original convection–diffusion Eq. (19). The second term on the right hand side of (29) represents the dissipation in the model (as it contains a second derivative) while the third term represents the model dispersion (as it contains the third derivative). Note that the dominant term on the right hand side (in the model terms) is the dissipation term. The stability of the model is governed only by the model dissipation (second derivative) term, which means that the second term on the right hand side of (29) must be nonnegative. Thus, we obtain the sub-characteristic condition

$$\lambda^2 \geqslant \{a(u)\}^2 \tag{30}$$

which connects the wave speed of the relaxation system to the wave speed of the convection–diffusion equation.

4. Finite Variable Difference Relaxation Scheme for viscous Burgers equation

4.1. Operator splitting

Each of the two Eqs. (17) and (18) of the relaxation system derived in the third section is solved using an operator splitting method, leading to convection–diffusion-source step and a relaxation step as follows:

Convection–diffusion-source step:

$$\frac{\partial f_1}{\partial t} - \lambda\frac{\partial f_1}{\partial x} = \frac{1}{2}v\frac{\partial^2 f_1}{\partial x^2} + \frac{1}{2}v\frac{\partial^2 f_2}{\partial x^2} \tag{31}$$
$$\frac{\partial f_2}{\partial t} + \lambda\frac{\partial f_2}{\partial x} = \frac{1}{2}v\frac{\partial^2 f_2}{\partial x^2} + \frac{1}{2}v\frac{\partial^2 f_1}{\partial x^2} \tag{32}$$

Relaxation step:

$$\frac{df_1}{dt} = -\frac{1}{\epsilon}[f_1 - F_1] \quad \text{and} \quad \frac{df_2}{dt} = -\frac{1}{\epsilon}[f_2 - F_2] \tag{33}$$

If we use an instantaneous relaxation to the equilibrium with $\epsilon = 0$ in the relaxation step, we obtain an *instantaneous relaxation step*:

$$f_1 = F_1 \quad \text{and} \quad f_2 = F_2 \tag{34}$$

Thus, we need to solve only Eqs. (31) and (32), with the above instantaneous relaxation step (34) as the restriction. We use it in the beginning of the time-step as

$$f_1^n = F_1^n \quad \text{and} \quad f_2^n = F_2^n \tag{35}$$

where n represents the time level t^n, the beginning of the time-step. Sakai's *Finite Variable Difference Method* (FVDM) is based on the exact solution of the steady linear convection–diffusion equation. We can obtain the steady linear convection diffusion equations by dropping unsteady and source terms from these two equations as

$$\frac{\partial^2 f_1}{\partial x^2} - \left(\frac{-2\lambda}{v}\right)\frac{\partial f_1}{\partial x} = 0 \tag{36}$$
$$\frac{\partial^2 f_2}{\partial x^2} + \left(\frac{-2\lambda}{v}\right)\frac{\partial f_2}{\partial x} = 0 \tag{37}$$

Note that the exact solution of such simpler convection–diffusion equations are used in the FVDM, but the final discretization includes all the source terms and unsteady (transient) terms, as will be shown in later subsections.

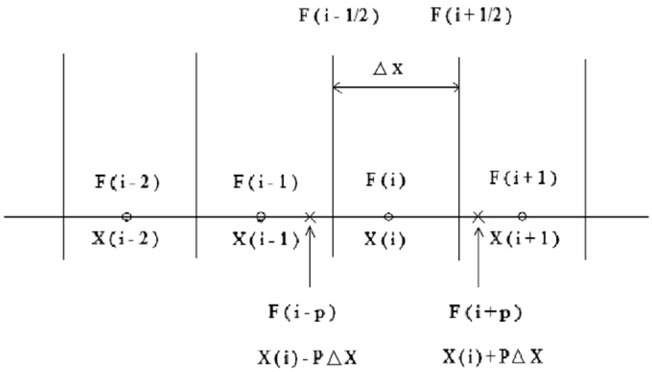


Fig. 1. Variable cell length in Finite Variable Difference Method.

4.2. Basics of Finite Variable Difference Method (FVDM)

In this section, we first present the basic idea of Sakai's Finite Variable Difference Method, as applied to Eqs. (36) and (37). Consider the convection diffusion Eq. (37)

$$\frac{\partial^2 f_2}{\partial x^2} - \left(\frac{2\lambda}{\nu}\right)\frac{\partial f_2}{\partial x} = 0 \tag{38}$$

where $\frac{2\lambda}{\nu} = R$ which is a constant and is like the Reynolds number in Navier–Stokes equations. The general solution to above equation is given as

$$f_2 = C_1 \exp[Rx] + C_2 \tag{39}$$

where C_1 and C_2 are constants determined by the boundary conditions. In Sakai's FVDM, a variable cell length is used in discretizing the first derivative $\frac{\partial f_2}{\partial x}$, as given below in (40) (see Fig. 1).

$$\frac{\partial f_2}{\partial x} = \frac{f_{2,i+p} - f_{2,i-p}}{2p\Delta x} \tag{40}$$

Here $f_{2,i+p}$ and $f_{2,i-p}$ are the quantities at $x = x_i + p\Delta x$, and $x = x_i - p\Delta x$ respectively, and they are approximated by the following upwind-biased differencing:

$$f_{2,i-p} = a_1 f_{2,i} + b_1 f_{2,i-1} + c_1 f_{2,i-2} \tag{41}$$
$$f_{2,i+p} = a_2 f_{2,i+1} + b_2 f_{2,i} + c_2 f_{2,i-1} \tag{42}$$

Here p is a kind of upwinding parameter. If we fix the coefficients in the above discretizations, then the FVDM is complete. To fix the coefficients, we fist need to fix the parameter p. It is worth noting that $p = \frac{1}{2}$ corresponds to the conventional finite difference method. If $p > \frac{1}{2}$, the weight of upwinding is larger. In Sakai's FVDM, the parameter p is fixed based on a *Locally Exact Numerical Scheme* in such a way that the variance of the solution is optimized. Such an optimized value of p is used in discretizing the space derivatives.

A few features of the upwinding parameter p are worth noting here. Conventional finite difference methods with $p = 0.5$ are not optimized for large mesh Reynolds numbers, greater than about 1, from the point of view of numerical stability and accuracy. Here, p is a kind of upwinding parameter and upwinding weight for discretizing the convection term in the FVDM is in substance larger than that in the conventional finite difference method. Therefore, the optimized FVDM keeps the numerical stability for all mesh Reynolds numbers as compared to the conventional finite difference method. This optimization highly improves the numerical accuracy for the linear (steady) convection–diffusion equation, giving a nonoscillatory solution, at even sufficiently large Reynolds numbers of upto 1000 [42].

4.3. Locally Exact Numerical Scheme

4.3.1. Difference coefficients

To evaluate the value of the variable at the cell interfaces (at $i \pm p\Delta x$), the following approximation is used:

$$\begin{aligned} f_2(x - p\Delta x) &= a_1 f_2(x) + b_1 f_2(x - \Delta x) + c_1 f_2(x - 2\Delta x) \\ f_2(x + p\Delta x) &= a_2 f_2(x + \Delta x) + b_2 f_2(x) + c_2 f_2(x - \Delta x) \end{aligned} \tag{43}$$

Applying Taylor series expansion on both the sides of Eq. (43) and comparing coefficients of $f_2(x)$ on both the sides we have

$$a_1 + b_1 + c_1 = 1 \quad \text{and} \quad a_2 + b_2 + c_2 = 1 \tag{44}$$

Following Sakai, we impose the condition that Eq. (43) should satisfy identically the exact solution of the convection–diffusion Eq. (39) for arbitrary values of C_1 and C_2. Assuming $C_1 = 1$ and $C_2 = 0$ we have

$$\begin{aligned} \exp[Rx_{i-p}] &= a_1 \exp[Rx_i] + b_1 \exp[Rx_{i-1}] + c_1 \exp[Rx_{i-2}] \\ \exp[Rx_{i+p}] &= a_2 \exp[Rx_{i+1}] + b_2 \exp[Rx_i] + c_2 \exp[Rx_{i-1}] \end{aligned} \tag{45}$$

We rewrite the above equation as

$$\begin{aligned} \exp[Rx_{i-p}] &= \exp[\ln a_1 + Rx_i] + \exp[\ln b_1 + Rx_{i-1}] + \exp[\ln c_1 + Rx_{i-2}] \\ \exp[Rx_{i+p}] &= \exp[\ln a_2 + Rx_{i+1}] + \exp[\ln b_2 + Rx_i] + \exp[\ln c_2 + Rx_{i-1}] \end{aligned}$$

Applying exponential series expansion to the above equation and comparing the coefficients of R on both sides, we have

$$\begin{aligned} x_{i-p} &= a_1 x_i + b_1 x_{i-1} + c_1 x_{i-2} \\ x_{i+p} &= a_2 x_{i+1} + b_2 x_i + c_2 x_{i-1} \end{aligned} \tag{46}$$

Writing Eqs. (44)–(46) together, we get the following matrix of equations for the difference coefficients:

$$[M]\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ x_{i-p} \\ \exp[Rx_{i-p}] \end{bmatrix} \tag{47}$$

$$[N]\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ x_{i+p} \\ \exp[Rx_{i+p}] \end{bmatrix} \tag{48}$$

where

$$[M] = \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_{i-1} & x_{i-2} \\ \exp[Rx_i] & \exp[Rx_{i-1}] & \exp[Rx_{i-2}] \end{bmatrix} \tag{49}$$

$$[N] = \begin{bmatrix} 1 & 1 & 1 \\ x_{i+1} & x_i & x_{i-1} \\ \exp[Rx_{i+1}] & \exp[Rx_i] & \exp[Rx_{i-1}] \end{bmatrix} \tag{50}$$

If p is given in Eqs. (47) and (48), we can obtain the coefficients $a_1, b_1, c_1, a_2, b_2, c_2$. Once these coefficients are available, they can be used in the approximation (43). Sakai evaluated the value of p by minimizing the variance of the solution. This procedure is explained for the equations considered here in the following subsections.

4.3.2. Characteristic equation

Consider now Eqs. (36) and (37). Discretizing the convection terms in Eq. (37) and using Eqs. (40)–(42), together with discretizing the diffusion terms with second order central differences, we obtain

$$\frac{f_{2,i+1} - 2f_{2,i} + f_{2,i-1}}{(\Delta x)^2} - R\frac{f_{2,i+p\Delta x} - f_{2,i-p\Delta x}}{2p\Delta x} = 0$$

or

$$(f_{2,i+1} - 2.f_{2,i} + f_{2,i-1}) - \left(\frac{R\Delta x}{2p}\right)\begin{bmatrix} (a_2 f_{2,i+1} + b_2 f_{2,i} + c_2 f_{2,i-1}) \\ -(a_1 f_{2,i} + b_1 f_{2,i-1} + c_1 f_{2,i-2}) \end{bmatrix} = 0$$

Let $R\Delta x = R'$, which is similar to the mesh Reynolds number. Rearranging the above equation yields the following difference equation:

$$Af_{2,i+1} + Bf_{2,i} + Cf_{2,i-1} + Df_{2,i-2} = 0 \tag{51}$$

where

$$\begin{aligned} A &= 1 - \tfrac{R'}{2p} a_2 \\ B &= -[2 + \tfrac{R'}{2p}(b_2 - a_1)] \\ C &= 1 - \tfrac{R'}{2p}(c_2 - b_1) \\ D &= \tfrac{R'}{2p} c_1 \end{aligned} \tag{52}$$

Eq. (51) has an exact solution (see [24]) given by

$$f_{2,i} = \alpha(Z_1)^i + \beta(Z_2)^i + \gamma(Z_3)^i \tag{53}$$

where α, β and γ are constants determined by the boundary conditions. In Eq. (53), Z_1, Z_2 and Z_3 are the roots of the characteristic equation given by

$$AZ^3 + BZ^2 + CZ + D = 0 \tag{54}$$

Since $a_1 + b_1 + c_1 = 1$ and $a_2 + b_2 + c_2 = 1$, from Eq. (52), we get the relation

$$A + B + C + D = -\left(\frac{R'}{2p}\right)[(a_2 + b_2 + c_2) - (a_1 + b_1 + c_1)] = 0 \tag{55}$$

Hence Eq. (54) has a root $Z_1 = 1$ and can be factorized as

$$(Z-1)\left\{\left[1 - \left(\frac{R'}{2p}\right)a_1\right]Z^2 - \left[1 + \left(\frac{R'}{2p}\right)(1 - c_2 - a_1)\right]Z - \left(\frac{R'}{2p}\right)c_1\right\} = 0 \tag{56}$$

From this equation, we obtain the other two roots as

$$Z_2 = \frac{1 + \left(\frac{R'}{2p}\right)(1 - c_2 - a_1) + \sqrt{\Sigma}}{2\left[1 - \left(\frac{R'}{2p}\right)a_2\right]} \tag{57}$$

$$Z_3 = \frac{1 + \left(\frac{R'}{2p}\right)(1 - c_2 - a_1) - \sqrt{\Sigma}}{2\left[1 - \left(\frac{R'}{2p}\right)a_2\right]} \tag{58}$$

where

$$\Sigma = \left[1 + \left(\frac{R'}{2p}\right)(1 - c_2 - a_1)\right]^2 + 4\left[1 - \left(\frac{R'}{2p}\right)a_2\right]\frac{R'}{2p}c_1 \tag{59}$$

4.3.3. Stability condition

Typically, the oscillations in the numerical solution occur because of the behaviour of the exact solution of the difference (discretized) equations. Therefore, it is reasonable to study the behaviour of the exact solution of the difference equation (instead of the round-off errors), by inspecting the positivity of roots of the difference equation, since the presence of a negative root indicates oscillations in the solution [41]. Therefore, we inspect the positivity of the roots (Z_1, Z_2, Z_3) for determining the stability of the scheme. Thus, we impose the condition as

$$\Sigma \geqslant 0, \quad Z_1 \geqslant 0, \quad Z_2 \geqslant 0, \quad Z_3 \geqslant 0. \tag{60}$$

We numerically examine the dependence of characteristic roots on $p\ (0.1 \leqslant p \leqslant 1)$ for $0.1 \leqslant R' \leqslant 1000$. The first stability condition $\Sigma \geqslant 0$ is always fulfilled for any p and R'. Both Z_2 and Z_3 are positive for all p under consideration in the case of $R' = 2$, while in the case of $R' = 10$ an asymptote $(p = p_a)$ for Z_2 exists, and Z_2 is negative for $p > p_a$. The asymptote $(p = p_a)$ for Z_2 occurs when the denominator of Eq. (57) is zero. The equation to determine p_a (for vanishing denominators in Eqs. (57) and (58)) is

$$1 - \left(\frac{R'}{2p_a}\right)a_2(p_a, R') = 0 \tag{61}$$

where the notation $a_2(p_a, R')$ is used, since the coefficient a_2 involves p and R' as parameters. When p approaches p_a, the numerator of Z_3 approaches zero. Hence Z_3 varies continuously even in the vicinity of $p = p_a$. A critical value R'_c, where Z_2 can be negative for R' greater than R'_c is given by Eq. (61) with $p_a = 1.0$, which is the maximum value of p. The equation to determine R'_c is

$$1 - \left(\frac{R'_c}{2}\right)a_2(1.0, R'_c) = 0 \tag{62}$$

Solving Eq. (62) numerically results in $R'_c = 2$. Then we solve Eq. (61) for $R' > R'_c$ and obtain the asymptote p_a in terms of R'. If p exceeds p_a, Z_2 becomes negative and the solution of Eq. (53) oscillates. Therefore, to get numerical stability, p must be

$$\begin{aligned} &(\text{for } R' \leqslant 2), \quad 0 < p < 1, \\ &(\text{for } R' > 2), \quad 0 < p < p_a = F(R') \end{aligned} \tag{63}$$

4.4. Variance of solution and optimum value of p

The variance σ is defined as

$$\sigma = \frac{1}{n}\sum_{i=1}^{n}[f_{2,i} - f_{2,i(e)}(x_i)]^2 \tag{64}$$

Here n is the total mesh number, $f_{2,i}$ and $f_{2,i(e)}(x_i)$ represent the numerical solution and the exact solution at the mesh number i, respectively. To evaluate σ , we perform typical calculations in one dimensional geometry with the uniform mesh $\Delta x = \frac{1}{n}$, in which the total mesh number n and the computational lengths are 15 and 1, respectively. The boundary values at $x = 0$ and $x = 1$ are set as $f_{2,i}(0) = 1$ and $f_{2,i}(1) = 0$. We have to determine value of p (denoted by p_0) which minimizes σ. Following Sakai, we tabulate the values of the variance for different possibilities and obtain the correlation equation of p_0 with respect to R' as

$$\begin{aligned}
&\text{for } (0 < R' \leqslant 14), \quad p_0 = G(R')\\
&\text{for } (14 \leqslant R' \leqslant 20), \quad p_0 = F(R') - 10^{-6}\ (20 - R')/6\\
&\text{for } (20 \leqslant R'), \quad p_0 = F(R')
\end{aligned} \tag{65}$$

The functions $G(R')$ and $F(R')$ can be obtained from the tabulated values of the variance (see [41]). Similar analysis can be done for Eq. (36), where R will be replaced by $-R$ and f_2 will be replaced by f_1.

4.5. Source term treatment, well-balancing and solution update

Let us now consider the equations to be solved in the convection–diffusion-source step: (31) and (32). To emphasize the concept of a *well-balancing*, let us first drop the unsteady terms in Eqs. (31) and (32), to obtain

$$\begin{aligned}
-\frac{\partial(\lambda f_1)}{\partial x} &= \frac{\partial\left(\frac{\nu}{2}\frac{\partial f_1}{\partial x}\right)}{\partial x} + \frac{\partial\left(\frac{\nu}{2}\frac{\partial f_2}{\partial x}\right)}{\partial x}\\
\frac{\partial(\lambda f_2)}{\partial x} &= \frac{\partial\left(\frac{\nu}{2}\frac{\partial f_2}{\partial x}\right)}{\partial x} + \frac{\partial\left(\frac{\nu}{2}\frac{\partial f_1}{\partial x}\right)}{\partial x}
\end{aligned} \tag{66}$$

Integrating above equations with respect to x we obtain

$$\begin{aligned}
-\lambda f_1 &= \frac{\nu}{2}\frac{\partial f_1}{\partial x} + \frac{\nu}{2}\frac{\partial f_2}{\partial x} + \text{constant}\\
\lambda f_2 &= \frac{\nu}{2}\frac{\partial f_2}{\partial x} + \frac{\nu}{2}\frac{\partial f_1}{\partial x} + \text{constant}
\end{aligned} \tag{67}$$

Adding both the equations, we get

$$\begin{aligned}
\lambda(f_2 - f_1) = \nu\frac{\partial f_1}{\partial x} + \nu\frac{\partial f_2}{\partial x} + \text{constant} \Rightarrow v = \nu\frac{\partial u}{\partial x} + \text{constant} \Rightarrow g(u) = g_\nu(u) + \text{constant}(\text{under relaxation limit})\\
\Rightarrow -g(u) + g_\text{v}(u) = \text{constant}
\end{aligned} \tag{68}$$

A numerical scheme that preserves the above steady state solution exactly at the cell-interfaces requires

$$-g(u_{i+p}) + g_\text{v}(u_{i+p}) = \text{constant}, \quad \forall i \tag{69}$$

or approximately with a formal second order accuracy

$$-g(u_{i+p}) + g_\text{v}(u_{i+p}) = \text{constant} + O((\Delta x)^2), \quad \forall i \tag{70}$$

Let us now consider the Finite Variable Difference Relaxation Scheme. Let $h_1 = -\lambda f_1$ and $h_2 = \lambda f_2$. Integrating Eqs. (31) and (32) over a finite volume $[x_{i-p}, x_{i+p}]$ and over a finite time interval $[t^n, t^{n+1}]$, we obtain

$$\int_{t^n}^{t^{n+1}}\int_{x_{i-p}}^{x_{i+p}}\frac{\partial f_1}{\partial t}\,dx\,dt + \int_{t^n}^{t^{n+1}}\int_{x_{i-p}}^{x_{i+p}}\frac{\partial h_1}{\partial x}\,dx\,dt = \int_{t^n}^{t^{n+1}}\int_{x_{i-p}}^{x_{i+p}}\frac{\nu}{2}\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}\right)dx\,dt + \int_{t^n}^{t^{n+1}}\int_{x_{i-p}}^{x_{i+p}}\frac{\nu}{2}\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial x}\right)dx\,dt \tag{71}$$

and

$$\int_{t^n}^{t^{n+1}}\int_{x_{i-p}}^{x_{i+p}}\frac{\partial f_2}{\partial t}\,dx\,dt + \int_{t^n}^{t^{n+1}}\int_{x_{i-p}}^{x_{i+p}}\frac{\partial h_2}{\partial x}\,dx\,dt = \int_{t^n}^{t^{n+1}}\int_{x_{i-p}}^{x_{i+p}}\frac{\nu}{2}\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial x}\right)dx\,dt + \int_{t^n}^{t^{n+1}}\int_{x_{i-p}}^{x_{i+p}}\frac{\nu}{2}\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}\right)dx\,dt \tag{72}$$

Thus, we obtain

$$\bar{f}_{1,i}^{n+1} = \bar{f}_{1,i}^{n} - \frac{\Delta t}{2p\Delta x}\left[h_{1,i+p}^n - h_{1,i-p}^n\right] + \frac{\nu}{2}\frac{\Delta t}{2p\Delta x}\left[\frac{\partial f_{1,i+p}^n}{\partial x} - \frac{\partial f_{1,i-p}^n}{\partial x}\right] + \frac{\nu}{2}\frac{\Delta t}{2p\Delta x}\left[\frac{\partial f_{2,i+p}^n}{\partial x} - \frac{\partial f_{2,i-p}^n}{\partial x}\right] \tag{73}$$

and

$$\bar{f}_{2,i}^{n+1} = \bar{f}_{2,i}^{n} - \frac{\Delta t}{2p\Delta x}\left[h_{2,i+p}^n - h_{2,i-p}^n\right] + \frac{\nu}{2}\frac{\Delta t}{2p\Delta x}\left[\frac{\partial f_{2,i+p}^n}{\partial x} - \frac{\partial f_{2,i-p}^n}{\partial x}\right] + \frac{\nu}{2}\frac{\Delta t}{2p\Delta x}\left[\frac{\partial f_{1,i+p}^n}{\partial x} - \frac{\partial f_{1,i-p}^n}{\partial x}\right] \tag{74}$$

The cell integral averages are defined by

$$\bar{f}_{1,i} = \frac{1}{2p\Delta x}\int_{x_{i-p}}^{x_{i+p}} f_1\,\mathrm{d}x \tag{75}$$

$$\bar{f}_{2,i} = \frac{1}{2p\Delta x}\int_{x_{i-p}}^{x_{i+p}} f_2\,\mathrm{d}x \tag{76}$$

Writing partial derivative terms as linear combination of neighbouring cell values, we have

$$\left(\frac{\partial f_1}{\partial x}\right)_{i+p} = a_3 f_{1,i+1} + b_3 f_{1,i} + c_3 f_{1,i-1} \tag{77}$$

$$\left(\frac{\partial f_1}{\partial x}\right)_{i-p} = a_4 f_{1,i} + b_3 f_{1,i-1} + c_3 f_{1,i-2} \tag{78}$$

Therefore, we obtain

$$\left(\frac{\partial f_1}{\partial x}\right)_{x+p\Delta x} = a_3 f_1(x+\Delta x) + b_3 f_1(x) + c_3 f_1(x-\Delta x)$$

$$\Rightarrow \frac{\partial f_1}{\partial x}(x) = a_3 f_1(x+(1-p)\Delta x) + b_3 f_1(x-p\Delta x) + c_3 f_1(x-(p+1)\Delta x) \tag{79}$$

Applying Taylor series expansion on both sides and comparing the coefficients of $f_1(x), \frac{\partial f_1(x)}{\partial x}$ and $\frac{\partial^2 f_1(x)}{\partial x^2}$ and solving system of three equations, we obtain

$$a_3 = \frac{2p+1}{2\Delta x}, \quad b_3 = \frac{-2p}{\Delta x}, \quad c_3 = \frac{1-2p}{\Delta x} \tag{80}$$

Similar analysis yields

$$a_4 = \frac{3-2p}{2\Delta x}, \quad b_4 = \frac{2(p-1)}{\Delta x}, \quad c_3 = \frac{1-2p}{2\Delta x} \tag{81}$$

In the same manner $\left(\frac{\partial f_2}{\partial x}\right)_{i+p}$ and $\left(\frac{\partial f_2}{\partial x}\right)_{i-p}$ can be evaluated. Note that $h_1 + h_2 = \lambda(f_2 - f_1) = v = g(u)$ (under relaxation limit). We can get the updating of the solution as

$$u_i^{n+1} = f_{1,i}^{n+1} + f_{2,i}^{n+1} \tag{82}$$

Under steady state conditions, Eqs. (73) and (74) become

$$-\left[h_{1,i+p}^n - h_{1,i-p}^n\right] + \frac{v}{2}\left[\frac{\partial f_{1,i+p}^n}{\partial x} - \frac{\partial f_{1,i-p}^n}{\partial x}\right] + \frac{v}{2}\left[\frac{\partial f_{2,i+p}^n}{\partial x} - \frac{\partial f_{2,i-p}^n}{\partial x}\right] = 0 \tag{83}$$

$$-\left[h_{2,i+p}^n - h_{2,i-p}^n\right] + \frac{v}{2}\left[\frac{\partial f_{2,i+p}^n}{\partial x} - \frac{\partial f_{2,i-p}^n}{\partial x}\right] + \frac{v}{2}\left[\frac{\partial f_{1,i+p}^n}{\partial x} - \frac{\partial f_{1,i-p}^n}{\partial x}\right] = 0 \tag{84}$$

Adding the above two equations, we obtain

$$-\left[\lambda f_{2,i+p}^n - \lambda f_{1,i+p}^n\right] + v\frac{\partial(f_{1,i+p}^n + f_{2,i+p}^n)}{\partial x} = -\left[\lambda f_{2,i-p}^n - \lambda f_{1,i-p}^n\right] + v\frac{\partial(f_{1,i-p}^n + f_{2,i-p}^n)}{\partial x} \tag{85}$$

$$\Rightarrow -g(u_{i+p}^n) + g_v(u_{i+p}^n) = -g(u_{i-p}^n) + g_v(u_{i-p}^n) \text{ (under relaxation limit)} \tag{86}$$

Hence this scheme is well balanced.

5. Finite Variable Difference Relaxation Scheme for 2D nonlinear conservation laws

As the relaxation system given by Jin and Xin is not diagonalizable in multi-dimensions, Aregba-Driollet and Natalini generalized the discrete Boltzmann equation in 1-D to multi-dimensions to obtain a multi-dimensional relaxation system as

$$\frac{\partial f}{\partial t} + \sum_{k=1}^{D} \Lambda_k \frac{\partial f}{\partial x_k} = \frac{1}{\epsilon}[F - f] \tag{87}$$

For the multi-dimensional diagonal relaxation system, the local Maxwellians are defined by

$$F_{D+1} = \frac{1}{D}\left[u + \frac{1}{\lambda}\sum_{k=1}^{D} g_k(u)\right]$$

$$F_i = -\frac{1}{\lambda}g_i(u) + F_{D+1}, \quad (i = 1, \ldots.., D) \tag{88}$$

The 2-D discrete Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + \Lambda_1 \frac{\partial f}{\partial x} + \Lambda_2 \frac{\partial f}{\partial y} = \frac{1}{\epsilon}[F - f] \tag{89}$$

where

$$\Lambda_1 = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \tag{90}$$

and

$$\Lambda_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \tag{91}$$

The local Maxwellians are defined by

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{u}{3} - \frac{2}{3\lambda} g_1(u) + \frac{1}{3\lambda} g_2(u) \\ \frac{u}{3} + \frac{1}{3\lambda} g_1(u) - \frac{2}{3\lambda} g_2(u) \\ \frac{u}{3} + \frac{1}{3\lambda} g_1(u) + \frac{1}{3\lambda} g_2(u) \end{bmatrix} \tag{92}$$

Expanding Eqs. (89) leads to the following equations:

$$\begin{aligned} &\tfrac{\partial f_1}{\partial t} - \lambda \tfrac{\partial f_1}{\partial x} = \tfrac{1}{\epsilon}[F_1 - f_1] \\ &\tfrac{\partial f_2}{\partial t} - \lambda \tfrac{\partial f_2}{\partial y} = \tfrac{1}{\epsilon}[F_2 - f_2] \\ &\tfrac{\partial f_3}{\partial t} + \lambda \tfrac{\partial f_3}{\partial x} + \lambda \tfrac{\partial f_3}{\partial y} = \tfrac{1}{\epsilon}[F_3 - f_3] \end{aligned} \tag{93}$$

where $u = f_1 + f_2 + f_3, g_1(u) = \lambda(f_3 - f_1)$ and $g_2 = \lambda(f_3 - f_2)$.

Consider 2-D viscous Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial g_1(u)}{\partial x} + \frac{\partial g_2(u)}{\partial y} = \frac{\partial g_{1v}(u)}{\partial x} + \frac{\partial g_{2v}(u)}{\partial y} \tag{94}$$

where $g_{1v}(u) = v \frac{\partial u}{\partial x}$ and $g_{2v} = v \frac{\partial u}{\partial y}$. Under relaxation approximation, as $\epsilon \to 0, f = F$. In analogy with the 1-D case, we introduce viscous terms in the discrete velocity Boltzmann equation in 2-D and utilize the following equations in the convection–diffusion-source step of a 2-D relaxation system.

$$\begin{aligned} &\frac{\partial f_1}{\partial t} - \lambda \frac{\partial f_1}{\partial x} = v \frac{\partial^2 f_1}{\partial x^2} + v \frac{\partial^2 f_2}{\partial x^2} \\ &\frac{\partial f_2}{\partial t} - \lambda \frac{\partial f_2}{\partial y} = v \frac{\partial^2 f_2}{\partial y^2} + v \frac{\partial^2 f_1}{\partial y^2} \\ &\frac{\partial f_3}{\partial t} + \lambda \frac{\partial f_3}{\partial x} + \lambda \frac{\partial f_3}{\partial y} = v \frac{\partial^2 f_3}{\partial x^2} + v \frac{\partial^2 f_3}{\partial y^2} \end{aligned} \tag{95}$$

Source terms in first two equations in the relaxation system (95) are treated using Jin's well-balanced scheme [23]. Note that the last equation contains no source terms. Integrating equations (95) over a finite volume with an area defined by $[x_{i-p}, x_{i+p}][y_{j-p}, y_{j+p}]$ (where p is a kind of upwind parameter) and over a finite time interval $[t^n, t^{n+1}]$, we obtain, after a little algebraic manipulation, the following expressions.

$$\bar{f}^{n+1}_{1,i,j} = \bar{f}^n_{1,i,j} + \frac{\Delta t}{2p\Delta x}[\lambda f^n_{1,i+p,j} - \lambda f^n_{1,i-p,j}] + v \frac{\Delta t}{2p\Delta x}\left[\frac{\partial f^n_{1,i+p,j}}{\partial x} - \frac{\partial f^n_{1,i-p,j}}{\partial x}\right] + v \frac{\Delta t}{2p\Delta x}\left[\frac{\partial f^n_{2,i+p,j}}{\partial x} - \frac{\partial f^n_{2,i-p,j}}{\partial x}\right] \tag{96}$$

$$\bar{f}^{n+1}_{2,i,j} = \bar{f}^n_{2,i,j} + \frac{\Delta t}{2p\Delta y}[\lambda f^n_{2,i,j+p} - \lambda f^n_{2,i,j-p}] + v \frac{\Delta t}{2p\Delta y}\left[\frac{\partial f^n_{2,i,j+p}}{\partial y} - \frac{\partial f^n_{2,i,j-p}}{\partial y}\right] + v \frac{\Delta t}{2p\Delta y}\left[\frac{\partial f^n_{1,i,j+p}}{\partial y} - \frac{\partial f^n_{1,i,j-p}}{\partial y}\right] \tag{97}$$

$$\bar{f}^{n+1}_{3,i,j} = \bar{f}^n_{3,i,j} - \frac{\Delta t}{2p\Delta x}[\lambda f^n_{3,i+p,j} - \lambda f^n_{3,i-p,j}] - \frac{\Delta t}{2p\Delta y}[\lambda f^n_{3,i,j+p} - \lambda f^n_{3,i,j-p}] + v \frac{\Delta t}{2p\Delta x}\left[\frac{\partial f^n_{3,i+p,j}}{\partial x} - \frac{\partial f^n_{3,i-p,j}}{\partial x}\right] + v \frac{\Delta t}{2p\Delta y}\left[\frac{\partial f^n_{3,i,j+p}}{\partial y} - \frac{\partial f^n_{3,i,j-p}}{\partial y}\right] \tag{98}$$

The solution can be recovered as

$$u^{n+1}_{i,j} = f^{n+1}_{1,i,j} + f^{n+1}_{2,i,j} + f^{n+1}_{3,i,j} \tag{99}$$

The cell interface fluxes are evaluated as in the 1-D case in each direction. The Finite Variable Difference Relaxation scheme, which is second order accurate, can yield mild wiggles near shocks, in the unsteady case. A simple minmax limiter is used to suppress the spurious oscillations produced by the second order scheme, near the discontinuities.

6. Extension of Finite Variable Difference Relaxation Scheme to shallow water equations

In this section, the Finite Variable Difference Relaxation Scheme is extended to the shallow water equations. Consider the 1-D shallow water equations, given by

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}=0 \tag{100}$$

$$\frac{\partial (hu)}{\partial t}+\frac{\partial\left(hu^2+\frac{1}{2}gh^2\right)}{\partial x}=-gh\frac{\partial B(x)}{\partial x}+\nu\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) \tag{101}$$

Since the Finite Variable Difference Method of Sakai is designed for convection–diffusion equations, we first introduce a fictitious viscosity term in the first equation of the above system (100) and take the value of the coefficient of fictitious viscosity (ν_f) to be a very low value, close to zero. With the fictitious viscosity term, the system can be written as

$$\frac{\partial h}{\partial t}+\frac{\partial (hu)}{\partial x}=\nu_f\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) \tag{102}$$

$$\frac{\partial (hu)}{\partial t}+\frac{\partial\left(hu^2+\frac{1}{2}gh^2\right)}{\partial x}=-gh\frac{\partial B(x)}{\partial x}+\nu\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) \tag{103}$$

The above system of equations can be written as

$$\frac{\partial U}{\partial t}+\frac{\partial G}{\partial x}=S(U) \tag{104}$$

where

$$U=\begin{bmatrix} h \\ hu \end{bmatrix},\quad G=\begin{bmatrix} hu \\ hu^2+\frac{1}{2}gh^2 \end{bmatrix} \tag{105}$$

and

$$S(U)=\begin{bmatrix} \nu_f\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) \\ -gh\frac{\partial B(x)}{\partial x}+\nu\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) \end{bmatrix} \tag{106}$$

We first introduce a *convection–gravity splitting* as

$$G=G^c+G^g \tag{107}$$

where

$$G^c=\begin{bmatrix} hu \\ hu^2 \end{bmatrix}\quad\text{and}\quad G^g=\begin{bmatrix} 0 \\ \frac{1}{2}gh^2 \end{bmatrix} \tag{108}$$

are the *convective flux* and the *gravity flux* respectively. Since the gravity essentially represents a force, we treat the *gravity flux* as a source term and include it in $S(U)$. With this modification, the 1-D shallow water equations are given by

$$\frac{\partial U}{\partial t}+\frac{\partial G^c}{\partial x}=\widetilde{S}(U) \tag{109}$$

where the modified source term which includes the gravity flux is now given by

$$\widetilde{S}(U)=\begin{bmatrix} \nu_f\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) \\ -\frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right)-gh\frac{\partial B}{\partial x}+\nu\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) \end{bmatrix} \tag{110}$$

A *relaxation system* for the above system of equations can now be introduced as

$$\frac{\partial U}{\partial t}+\frac{\partial V}{\partial x}=\widetilde{S}(U) \tag{111}$$

$$\frac{\partial V}{\partial t}+D\frac{\partial U}{\partial x}=-\frac{1}{\epsilon}\left[V-G^c\right],\quad \epsilon\to 0 \tag{112}$$

where D is a diagonal matrix, defined by

$$D=\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \tag{113}$$

For simplicity, let us assume $\lambda_1^2=-\lambda_2^2=\lambda^2$ and the value of λ is taken to the be the global maximum of the absolute values of the eigenvalues of the flux Jacobian matrix $A^c=\frac{\partial G^c}{\partial U}$. The quasi-linear form of the relaxation system is given by

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} = H \tag{114}$$

where

$$Q = \begin{bmatrix} U \\ V \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ D & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} \widetilde{S}(U) \\ -\frac{1}{\epsilon}(V - G^c) \end{bmatrix} \tag{115}$$

Since the convection terms on the left hand side of the relaxation system represent the hyperbolic part, we can introduce characteristic variables and decouple the system (as in Section 2) to obtain the following two equations:

$$\frac{\partial f_1}{\partial t} - \lambda\frac{\partial f_1}{\partial x} = \frac{1}{2}\widetilde{S}(U) + \frac{1}{2\lambda\epsilon}[V - G^c] \tag{116}$$

$$\frac{\partial f_2}{\partial t} + \lambda\frac{\partial f_2}{\partial x} = \frac{1}{2}\widetilde{S}(U) - \frac{1}{2\lambda\epsilon}[V - G^c] \tag{117}$$

where

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}U - \frac{1}{2\lambda}V \\ \frac{1}{2}U + \frac{1}{2\lambda}V \end{bmatrix} \tag{118}$$

Introducing the new variable F as

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}U - \frac{1}{2\lambda}G^c \\ \frac{1}{2}U + \frac{1}{2\lambda}G^c \end{bmatrix} \tag{119}$$

and using the definition $U = f_1 + f_2$, we obtain

$$\frac{\partial f_1}{\partial t} - \lambda\frac{\partial f_1}{\partial x} = \frac{1}{2}\widetilde{S}(U) - \frac{1}{\epsilon}[f_1 - F_1] \tag{120}$$

$$\frac{\partial f_2}{\partial t} + \lambda\frac{\partial f_2}{\partial x} = \frac{1}{2}\widetilde{S}(U) - \frac{1}{\epsilon}[f_2 - F_2] \tag{121}$$

where f_1 and f_2 are vectors with two components each for 1-D case. Note that each of f_1 and f_2 contains two components.

$$f_1 = \begin{bmatrix} f_{1,1} \\ f_{1,2} \end{bmatrix} \quad \text{and} \quad f_2 = \begin{bmatrix} f_{2,1} \\ f_{2,2} \end{bmatrix} \tag{122}$$

Therefore, after substituting values of $\widetilde{S}(U)$, we get following set of decoupled equations:

$$\frac{\partial f_{1,1}}{\partial t} - \lambda\frac{\partial f_{1,1}}{\partial x} = \frac{1}{2}v_f\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) - \frac{1}{\epsilon}[f_{1,1} - F_{1,1}] \tag{123}$$

$$\frac{\partial f_{1,2}}{\partial t} - \lambda\frac{\partial f_{1,2}}{\partial x} = \frac{1}{2}\left[-\frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right) - gh\frac{\partial B}{\partial x} + v\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)\right] - \frac{1}{\epsilon}[f_{1,2} - F_{1,2}] \tag{124}$$

$$\frac{\partial f_{2,1}}{\partial t} + \lambda\frac{\partial f_{2,1}}{\partial x} = \frac{1}{2}v_f\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) - \frac{1}{\epsilon}[f_{2,1} - F_{2,1}] \tag{125}$$

$$\frac{\partial f_{2,2}}{\partial t} + \lambda\frac{\partial f_{2,2}}{\partial x} = \frac{1}{2}\left[-\frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right) - gh\frac{\partial B}{\partial x} + v\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)\right] - \frac{1}{\epsilon}[f_{2,2} - F_{2,2}] \tag{126}$$

Sakai's Finite Variable Difference Method is applied to the aforementioned system of equations by converting the fictitious viscous term and viscous term (corresponding to mass and momentum conservation equations respectively) in terms of characteristic variables. From $U = f_1 + f_2$, we get

$$h = f_{1,1} + f_{2,1} \quad \text{and} \quad hu = f_{1,2} + f_{2,2}$$

Therefore

$$v_f\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = v_f\frac{\partial}{\partial x}\left(\frac{\partial(f_{1,1} + f_{2,1})}{\partial x}\right) = v_f\frac{\partial^2 f_{1,1}}{\partial x^2} + v_f\frac{\partial^2 f_{2,1}}{\partial x^2} \tag{127}$$

and

$$\frac{\partial}{\partial x}(hu) = \frac{\partial f_{1,2}}{\partial x} + \frac{\partial f_{2,2}}{\partial x}$$

$$\text{or } h\frac{\partial u}{\partial x} + u\frac{\partial h}{\partial x} = \frac{\partial f_{1,2}}{\partial x} + \frac{\partial f_{2,2}}{\partial x}$$

Differentiating both sides with respect to x and multiplying by v we obtain

$$
\begin{aligned}
& v\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)+v\frac{\partial}{\partial x}\left(u\frac{\partial h}{\partial x}\right)=v\frac{\partial^2 f_{1,2}}{\partial x^2}+v\frac{\partial^2 f_{2,2}}{\partial x^2}\\
\text{or } & v\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)=v\frac{\partial^2 f_{1,2}}{\partial x^2}+v\frac{\partial^2 f_{2,2}}{\partial x^2}-v\frac{\partial}{\partial x}\left(u\frac{\partial h}{\partial x}\right)\\
\text{or } & v\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)=v\frac{\partial^2 f_{1,2}}{\partial x^2}+v\frac{\partial^2 f_{2,2}}{\partial x^2}-v\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial}{\partial x}(f_{1,1}+f_{2,1})\right)\\
\text{or } & v\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)=v\frac{\partial^2 f_{1,2}}{\partial x^2}+v\frac{\partial^2 f_{2,2}}{\partial x^2}-v\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{1,1}}{\partial x}\right)-v\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{2,1}}{\partial x}\right)
\end{aligned}
\tag{128}
$$

where

$$
\frac{U_2}{U_1}=u=\frac{hu}{h}=\frac{f_{1,2}+f_{2,2}}{f_{1,1}+f_{2,1}}
\tag{129}
$$

Substituting (127) in (123), (125) and (128) in (124), (126), we obtain

$$
\frac{\partial f_{1,1}}{\partial t}-\lambda\frac{\partial f_{1,1}}{\partial x}=\frac{1}{2}v_f\frac{\partial^2 f_{1,1}}{\partial x^2}+\frac{1}{2}v_f\frac{\partial^2 f_{2,1}}{\partial x^2}-\frac{1}{\epsilon}[f_{1,1}-F_{1,1}]
\tag{130}
$$

$$
\begin{aligned}
\frac{\partial f_{1,2}}{\partial t}-\lambda\frac{\partial f_{1,2}}{\partial x}=&\frac{1}{2}v\frac{\partial^2 f_{1,2}}{\partial x^2}+\frac{1}{2}v\frac{\partial^2 f_{2,2}}{\partial x^2}-\frac{1}{2}v\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{1,1}}{\partial x}\right)-\frac{1}{2}v\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{2,1}}{\partial x}\right)\\
&+\frac{1}{2}\left[-\frac{\partial}{\partial x}\left(\frac{1}{2}g(f_{1,1}+f_{2,1})^2\right)-g(f_{1,1}+f_{2,1})\frac{\partial B}{\partial x}\right]-\frac{1}{\epsilon}[f_{1,2}-F_{1,2}]
\end{aligned}
\tag{131}
$$

$$
\frac{\partial f_{2,1}}{\partial t}+\lambda\frac{\partial f_{2,1}}{\partial x}=\frac{1}{2}v_f\frac{\partial^2 f_{1,1}}{\partial x^2}+\frac{1}{2}v_f\frac{\partial^2 f_{2,1}}{\partial x^2}-\frac{1}{\epsilon}[f_{2,1}-F_{2,1}]
\tag{132}
$$

$$
\begin{aligned}
\frac{\partial f_{2,2}}{\partial t}+\lambda\frac{\partial f_{2,2}}{\partial x}=&\frac{1}{2}v\frac{\partial^2 f_{1,2}}{\partial x^2}+\frac{1}{2}v\frac{\partial^2 f_{2,2}}{\partial x^2}-\frac{1}{2}v\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{1,1}}{\partial x}\right)-\frac{1}{2}v\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{2,1}}{\partial x}\right)\\
&+\frac{1}{2}\left[-\frac{\partial}{\partial x}\left(\frac{1}{2}g(f_{1,1}+f_{2,1})^2\right)-g(f_{1,1}+f_{2,1})\frac{\partial B}{\partial x}\right]-\frac{1}{\epsilon}[f_{2,2}-F_{2,2}]
\end{aligned}
\tag{133}
$$

Defining $\tilde{v}_f=\frac{1}{2}v_f$ and $\tilde{v}=\frac{1}{2}v$, we can rewrite Eqs. (130) to (133) as

$$
\frac{\partial f_{1,1}}{\partial t}-\lambda\frac{\partial f_{1,1}}{\partial x}=\tilde{v}_f\frac{\partial^2 f_{1,1}}{\partial x^2}-\frac{1}{\epsilon}[f_{1,1}-F_{1,1}]+\bar{S}_{1,1}
\tag{134}
$$

$$
\frac{\partial f_{1,2}}{\partial t}-\lambda\frac{\partial f_{1,2}}{\partial x}=\tilde{v}_f\frac{\partial^2 f_{1,2}}{\partial x^2}-\frac{1}{\epsilon}[f_{1,2}-F_{1,2}]+\bar{S}_{1,2}
\tag{135}
$$

$$
\frac{\partial f_{2,1}}{\partial t}+\lambda\frac{\partial f_{2,1}}{\partial x}=\tilde{v}_f\frac{\partial^2 f_{2,1}}{\partial x^2}-\frac{1}{\epsilon}[f_{2,1}-F_{2,1}]+\bar{S}_{2,1}
\tag{136}
$$

$$
\frac{\partial f_{2,2}}{\partial t}+\lambda\frac{\partial f_{2,2}}{\partial x}=\tilde{v}_f\frac{\partial^2 f_{2,2}}{\partial x^2}-\frac{1}{\epsilon}[f_{2,2}-F_{2,2}]+\bar{S}_{2,2}
\tag{137}
$$

where the source terms in the above equations are given by

$$
\begin{aligned}
\bar{S}_{1,1}&=\tilde{v}_f\frac{\partial^2 f_{2,1}}{\partial x^2}\\
\bar{S}_{1,2}&=\tilde{v}\frac{\partial^2 f_{2,2}}{\partial x^2}-\tilde{v}\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{1,1}}{\partial x}\right)-\tilde{v}\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{2,1}}{\partial x}\right)-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{2}g(f_{1,1}+f_{2,1})^2\right)-\frac{1}{2}g(f_{1,1}+f_{2,1})\frac{\partial B}{\partial x}\\
\bar{S}_{2,1}&=\tilde{v}_f\frac{\partial^2 f_{1,1}}{\partial x^2}\\
\bar{S}_{2,2}&=\tilde{v}\frac{\partial^2 f_{1,2}}{\partial x^2}-\tilde{v}\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{1,1}}{\partial x}\right)-\tilde{v}\frac{\partial}{\partial x}\left(\frac{U_2}{U_1}\frac{\partial f_{2,1}}{\partial x}\right)-\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{2}g(f_{1,1}+f_{2,1})^2\right)-\frac{1}{2}g(f_{1,1}+f_{2,1})\frac{\partial B}{\partial x}
\end{aligned}
\tag{138}
$$

Note that each of Eqs. (134), (137) is a convection–diffusion equation together with a relaxation term and a source term. We apply the Finite Variable Difference Method of Sakai to each of these convection–diffusion equations, treating the source terms using the strategy introduced by Jin (as in Section 4). Note also that the final convection–diffusion equations used are linear while source terms in both the equations are nonlinear and both FVDM of Sakai and Jin's strategy for discretizing the source term are well-suited for these terms, respectively. The value of fictitious viscosity coefficient in modified mass conservation equation is taken as $v_f=10^{-4}$. For the case of 2-D shallow water equations, the convection–gravity splitting method is similar to the 1-D case as explained before. Together with this strategy of treating the gravity terms as source terms, the 2-D discrete velocity Boltzmann equation (introduced in Section 5) is utilized as the relaxation system and the FVDM for the resulting 2-D convection–diffusion equations is as given in Section 5.

7. Numerical experiments

This new algorithm, Finite Variable Difference Relaxation Scheme, is tested on several benchmark problems for inviscid and viscous Burgers equations and for shallow water equations both in one and two dimensions.

7.1. 1-D inviscid Burgers equation test case

This test case is taken from Laney [25] and models an expansion wave and a shock wave. The new scheme is applied to this problem and is compared with the exact solution (see Fig. 2). The coefficient of viscosity is taken to be very small in order to obtain the vanishing viscosity results. The time-step is evaluated using the CFL condition based on the maximum of the eigenvalues of the flux Jacobian matrix for the convection terms. Since we are dealing with vanishing viscosity limits, this strategy is sufficient. For viscous terms with larger viscosity coefficients, in the evaluation of the time-step, the effect of diffusion terms also needs to be taken care of. The parameter λ is chosen as the absolute value of the maximum of the eigen-

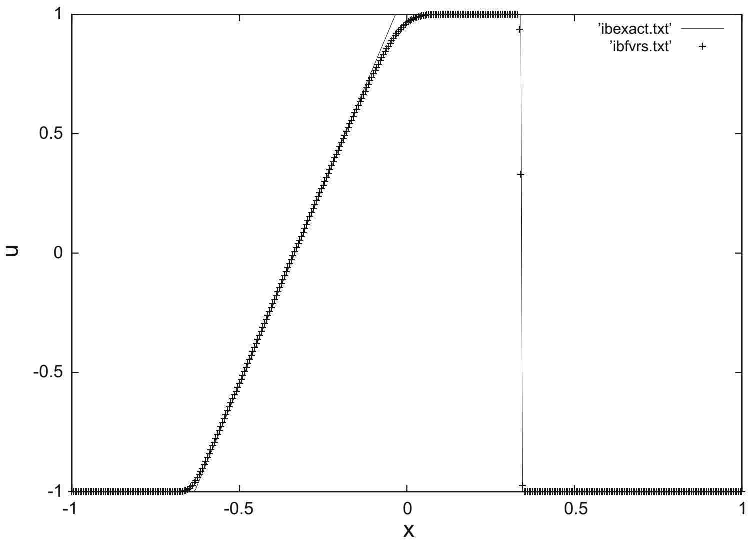


Fig. 2. 1-D inviscid Burgers equation solution with Finite Variable Difference Relaxation Scheme; $\Delta x = 0.01$.

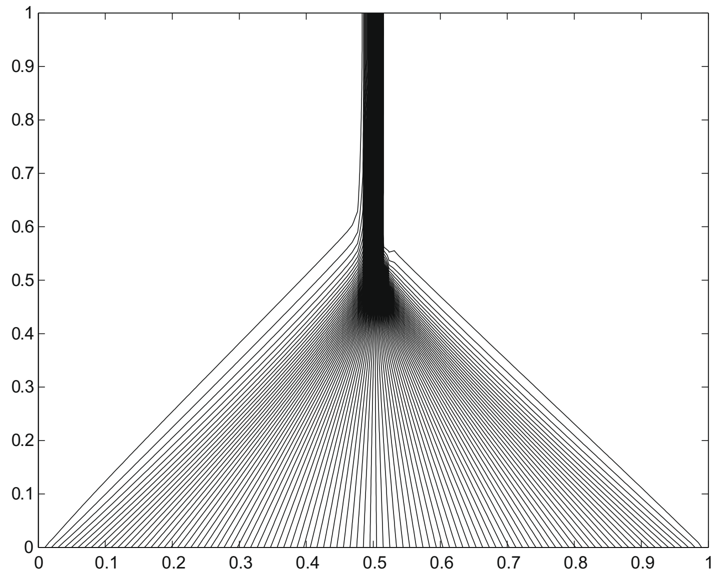


Fig. 3. 2-D inviscid Burgers equation solution (normal shock case) with Finite Variable Difference Relaxation Scheme; $\Delta x = \Delta y = \frac{1}{128}$.

values of the flux Jacobian matrix in the domain, by mimicing the sub-characteristic condition in Eq. (30) for the relevant hyperbolic system of equations. We can see very accurate resolution of the expansion wave and the shock wave in this test case.

7.2. 2-D inviscid Burgers equation test cases

Problem: On the square $[0, 1] \times [0, 1]$, we consider the nonlinear problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} = 0$$

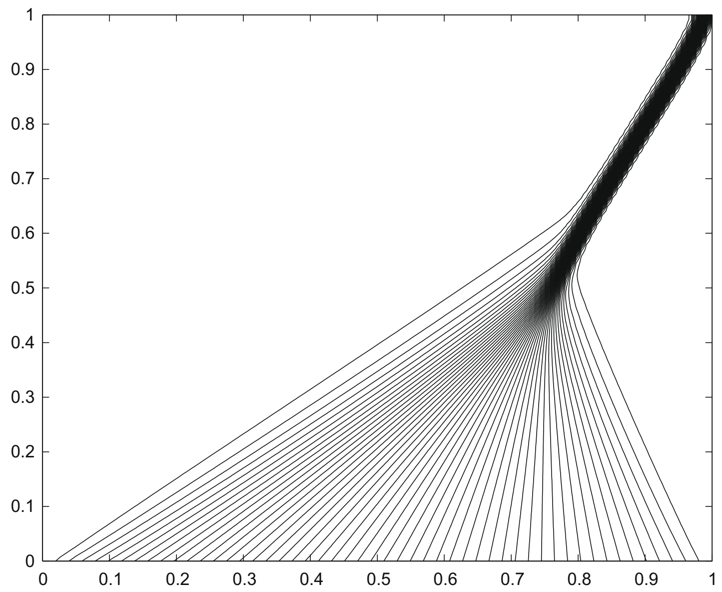


Fig. 4. 2-D inviscid Burgers equation solution (oblique shock case) with Finite Variable Difference Relaxation Scheme; $\Delta x = \Delta y = \frac{1}{128}$.

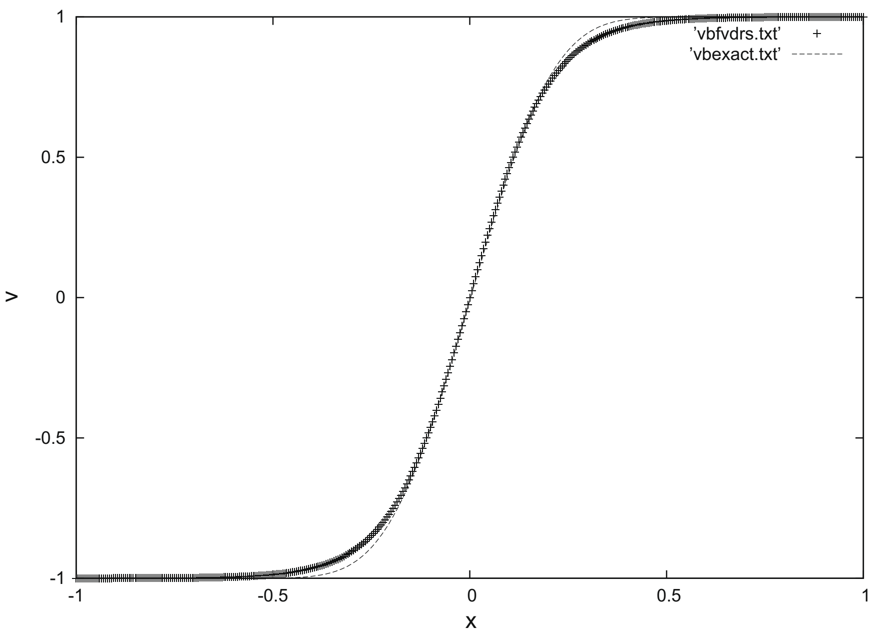


Fig. 5. Steady state solution of 1-D viscous Burgers equation with Finite Variable Difference Relaxation Scheme; $\nu = 0.1$; $\Delta x = 0.01$.

where

$$f(u) = \frac{1}{2}u^2, \quad g(u) = u$$

Two sets of boundary conditions have been considered:

Case 1. Boundary conditions:

$$\begin{aligned} &u(0,y) = 1, \quad 0 < y < 1 \\ &u(1,y) = -1, \quad 0 < y < 1 \\ &u(x,0) = 1 - 2x, \quad 0 < x < 1 \end{aligned} \tag{139}$$

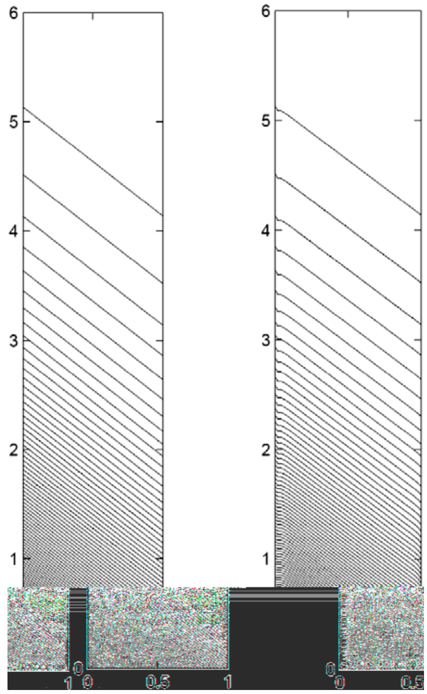


Fig. 6. 2-D viscous Burgers equation test case (left) initial condition and (right) steady state solution; $\Delta x = \Delta y = 0.02$.

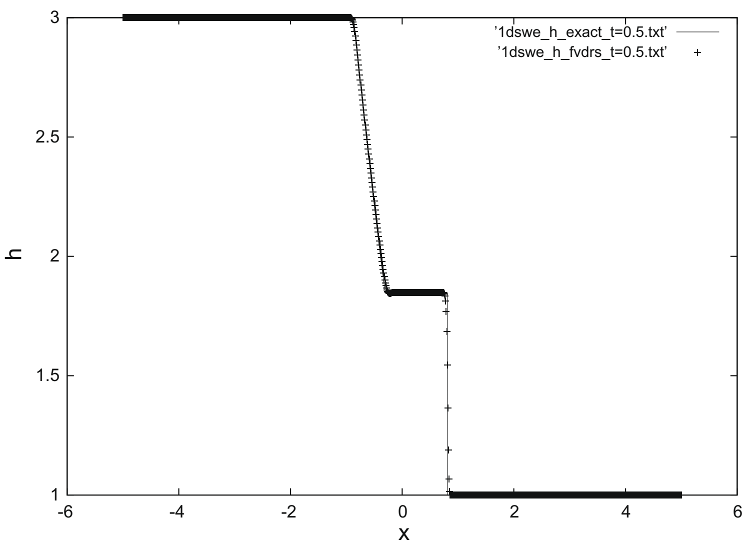


Fig. 7. Solution of depth h (+) at $t = 0.5$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

On the top boundary, Neumann boundary conditions are applied. This test case is taken from [45], as well as the next one. It models a shock wave and a smooth variation representing an expansion fan in a 2-D domain. Both features are captured very well by the Finite Variable Difference Relaxation Scheme (see Figs. 3, 4).

Case 2. Boundary conditions:

$$\begin{aligned} &u(0,y) = 1.5, \quad 0 < y < 1 \\ &u(1,y) = -0.5, \quad 0 < y < 1 \\ &u(x,0) = 1.5 - 2x, \quad 0 < x < 1 \end{aligned} \tag{140}$$

7.3. 1-D viscous Burgers equation test case

For the viscous Burgers equation diffusive effects can be expected near the discontinuities. Numerical solution obtained by the Finite Variable Difference Relaxation Scheme at steady state is compared with the exact solution for the 1-D viscous Burgers equation given by

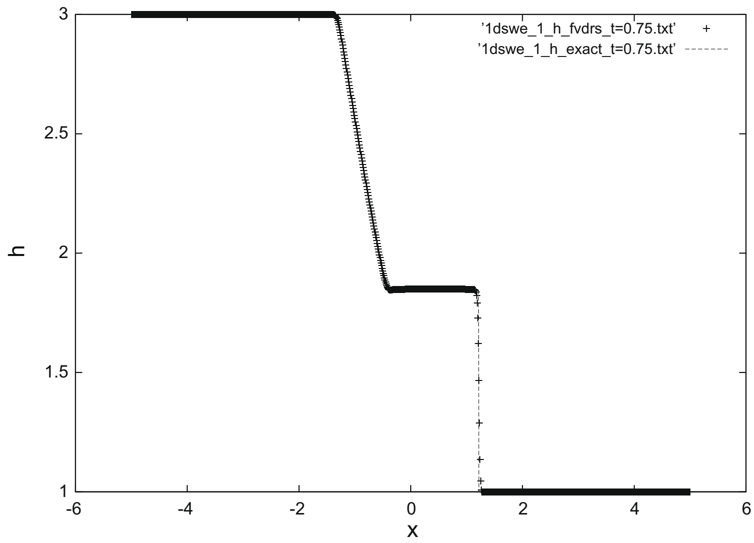


Fig. 8. Solution of depth h (+) at $t = 0.75$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

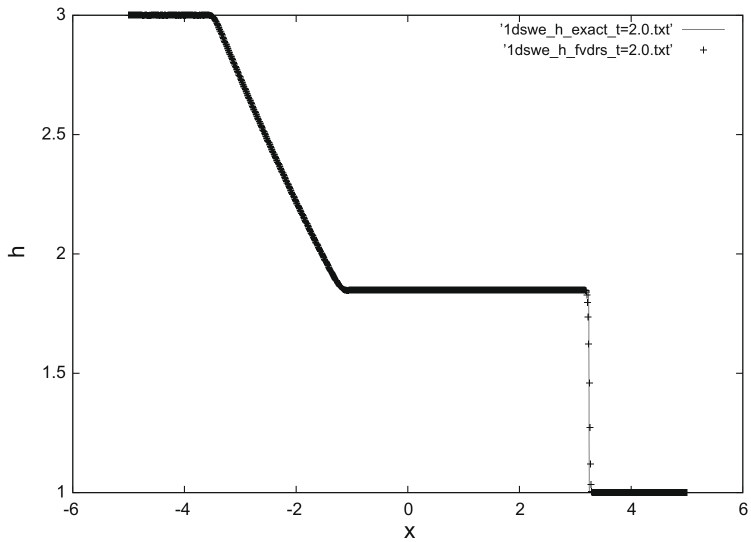


Fig. 9. Solution of depth h (+) at $t = 2$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = \frac{\partial g_v(u)}{\partial x}$$

where $g(u) = \frac{1}{2}u^2$ and $g_v(u) = v\frac{\partial u}{\partial x}$, subject to the conditions

$$\begin{aligned} u &= -1, \quad x \leqslant 0 \\ u &= 1, \quad x > 0 \end{aligned} \tag{141}$$

The solution for the above1-D viscous Burgers equation in steady state, *i.e.*, for the following equation:

$$u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2} \tag{142}$$

for the initial condition

$$u = \begin{cases} u_L \text{ at } x = -\infty \\ u_R \text{ at } x = +\infty \end{cases} \tag{143}$$

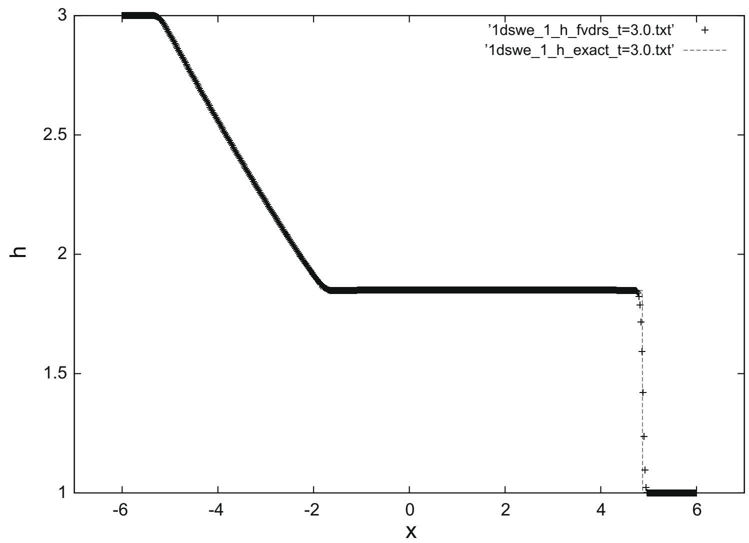


Fig. 10. Solution of depth h (+) at $t = 3$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

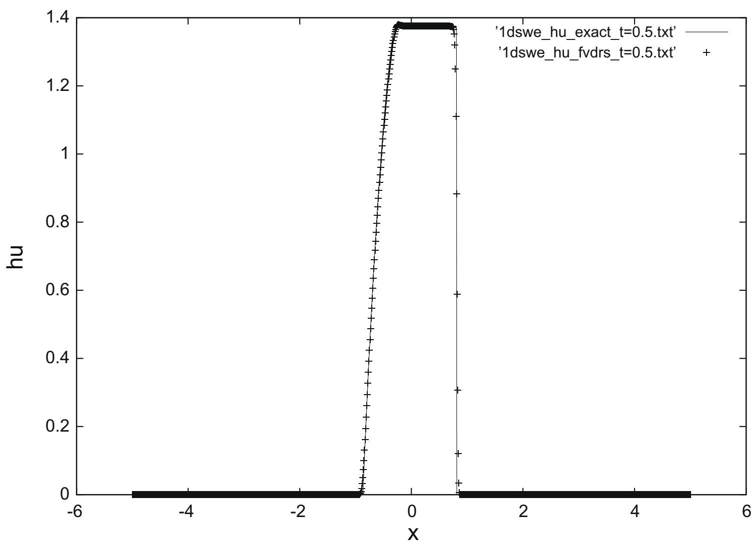


Fig. 11. Solution of momentum hu (+) at $t = 0.5$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

is given by [9]

$$u = \frac{1}{2}\left[(u_L + u_R) - (u_L - u_R)\tanh\left[\frac{x(u_L - u_R)}{4v}\right]\right] \tag{144}$$

The results show good resolution of the solution with very little numerical dissipation as compared to the exact solution (see Fig. 5).

7.4. 2-D viscous Burgers equation test case

This test case is taken from [30]. We consider viscous Burgers equation as

$$\frac{\partial u}{\partial t} = -u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + 0.01\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

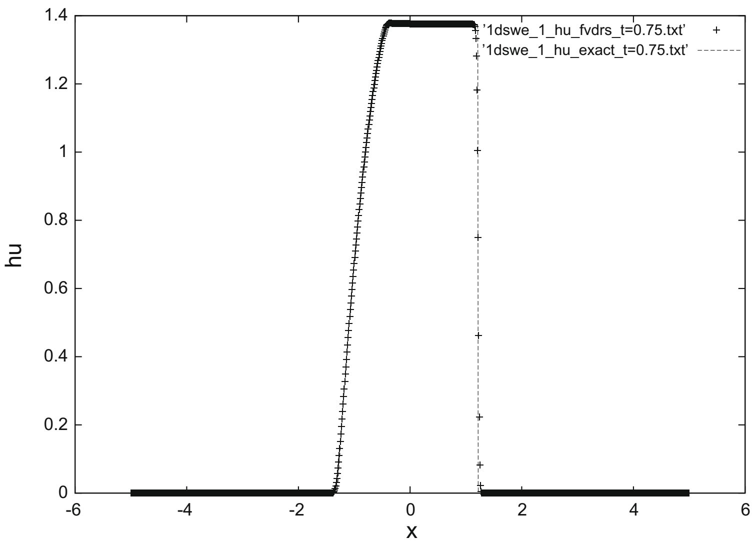


Fig. 12. Solution of momentum hu (+) at $t = 0.75$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

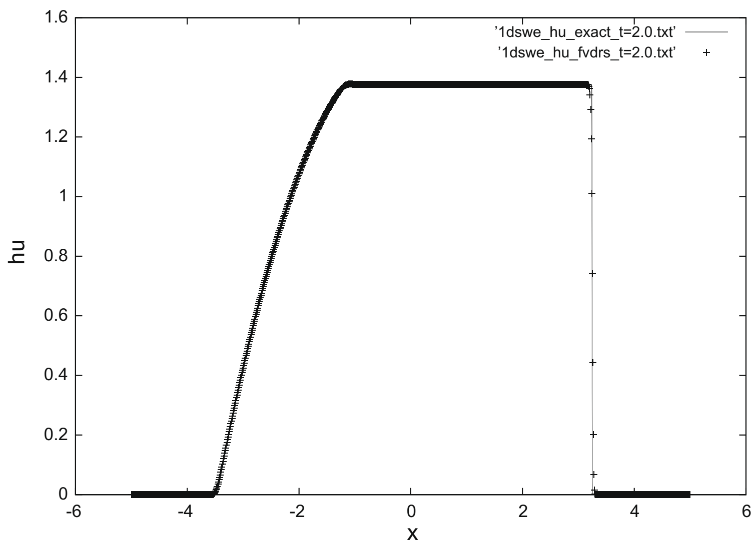


Fig. 13. Solution of momentum hu (+) at $t = 2$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

We take the initial and boundary conditions of the form

$$1 + \exp(x + y - t)^{-1} \tag{145}$$

With these initial conditions, the solution is a straight line wave (u is constant for $x = -y$) moving in the direction $\theta = \frac{\pi}{4}$. The numerical solution obtained is well in agreement with the exact solution (see Fig. 6).

7.5. 1-D shallow water equations test cases

Consider "Viscous Saint-Venant system" for shallow water equations with topography neglecting bed friction [33], as

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \tag{146}$$

$$\frac{\partial hu}{\partial t} + \frac{\partial \left(hu^2\right)}{\partial x} + \frac{\partial \left(\frac{1}{2} g h^2\right)}{\partial x} = -gh\frac{\partial B(x)}{\partial x} + \nu \frac{\partial}{\partial x} h \left(\frac{\partial u}{\partial x}\right) \tag{147}$$

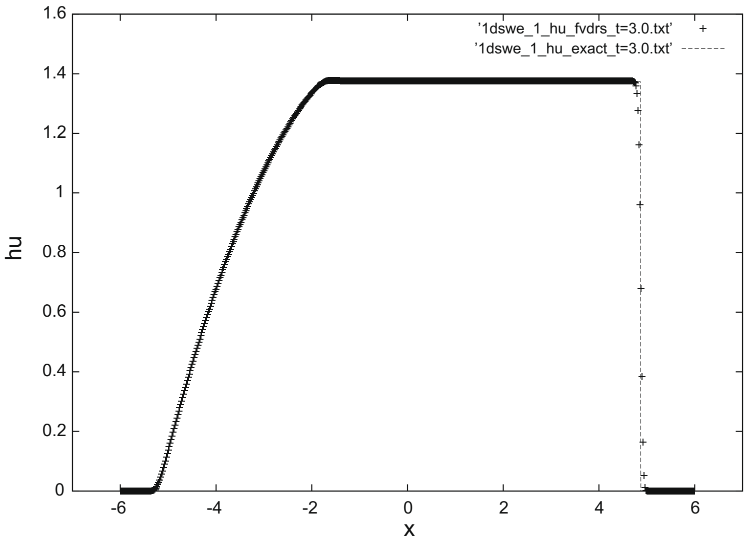


Fig. 14. Solution of momentum hu (+) at $t = 3$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

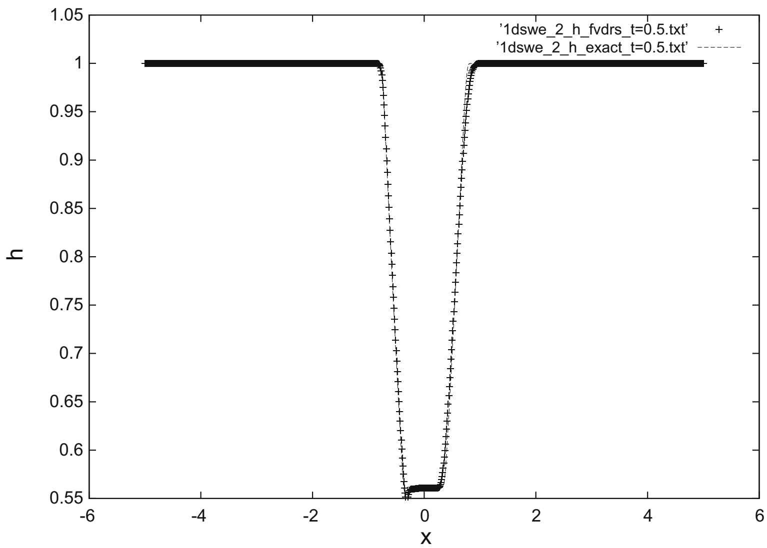


Fig. 15. Solution of depth h (+) at $t = 0.5$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

Here, h is depth of the water, u is the mean velocity, g is the gravitational constant, ν is the coefficient of viscosity and $B(x)$ is the bottom elevation. The Finite Variable Difference Relaxation Scheme (FVDRS) presented in Section 6 is applied under vanishing viscosity limits for several benchmark problems. Spurious oscillations in the vicinity of the shocks are suppressed by using a minmod limiter. The 1-d shallow water equations are tested on the following bench-mark problems.

7.5.1. Dam break flow

This test case is taken from [27]. Consider the shallow water equations with the piecewise constant initial data as

$$h(x,0) = 3, \quad u(x,0) = 0, \quad \text{for } x \leqslant 0 \tag{148}$$
$$h(x,0) = 1, \quad u(x,0) = 0, \quad \text{for } x > 0 \tag{149}$$

The spatial domain is considered initially as $[-5, 5]$, which is extended to one of the solutions at a later time as $[-6, 6]$. Solution of the above dam-break problem is shown at times $t = 0.5, t = 0.75$, $t = 2.0$ and $t = 3.0$ in Figs. 7–10 for depth h and in Figs. 11–14 for momentum hu, with $\Delta x = 0.01$. The exact solution is provided for comparison.

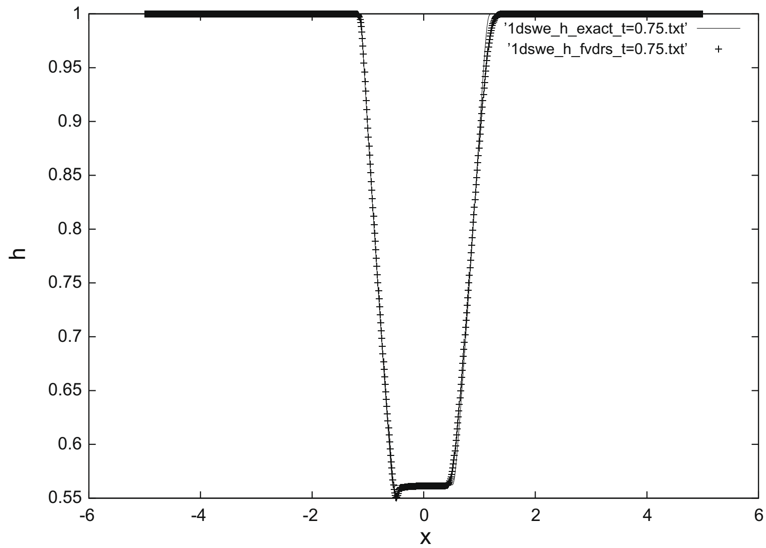


Fig. 16. Solution of depth h (+) at $t = 0.75$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

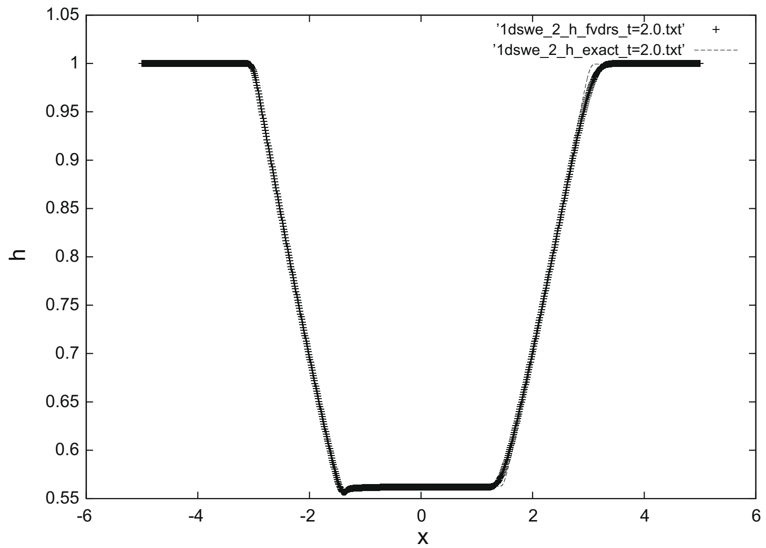


Fig. 17. Solution of depth h (+) at $t = 2$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

7.5.2. Rarefaction riemann problem

This test case is taken from [27]. Consider the Riemann problem for the shallow water equations with data given as follows:

$$h(x,0) = 1.0, \quad u(x,0) = -0.5, \quad \text{for } x \leqslant 0 \tag{150}$$

$$h(x,0) = 1.0, \quad u(x,0) = 0.5, \quad \text{for } x > 0 \tag{151}$$

The spatial domain is taken initially as $[-5,5]$. Solution of the above dam-break problem is shown at times $t = 0.5, t = 0.75, t = 2.0$ and $t = 3.0$ in Figs. 15–18 for depth h and in Figs. 19–22 for momentum hu, with $\Delta x = 0.01$. The exact solution is provided for comparison.

7.5.3. Riemann problem with bottom topography

This test case is taken from [23]. Consider Riemann initial data given by

$$h(x,0) = 1, \quad u(x,0) = 0, \quad \text{for } x \leqslant 0 \tag{152}$$

$$h(x,0) = 0.2, \quad u(x,0) = 0, \quad \text{for } x > 0 \tag{153}$$

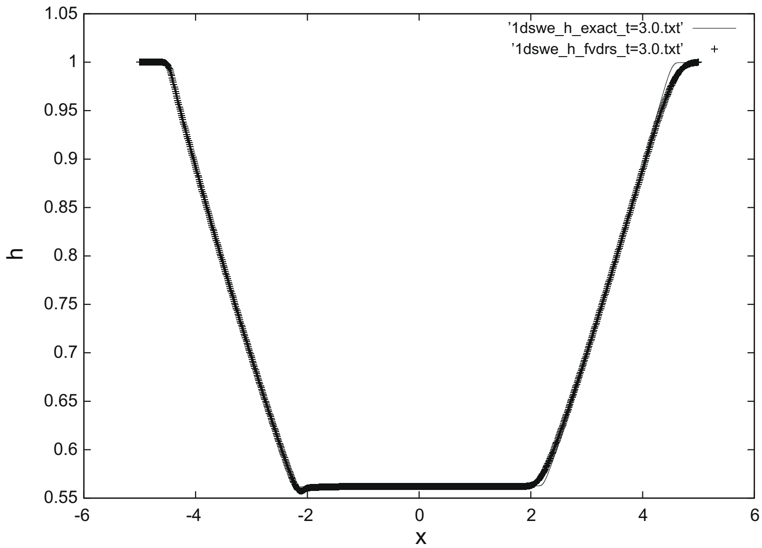


Fig. 18. Solution of depth h (+) at $t = 3$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

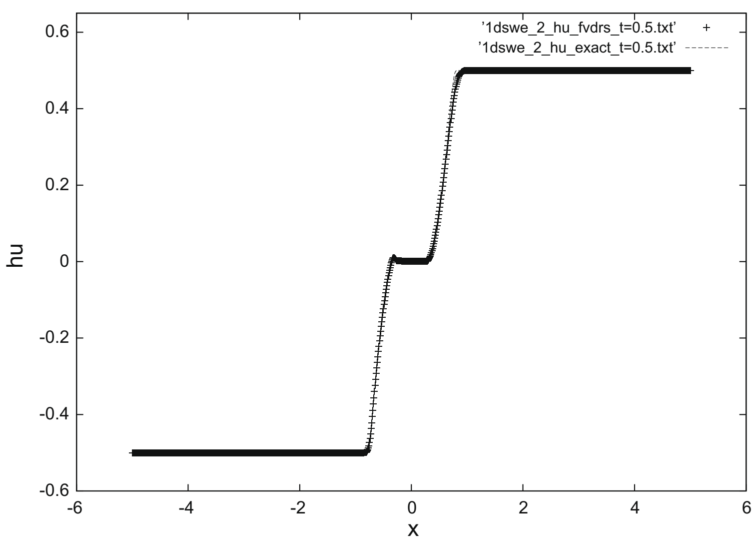


Fig. 19. Solution of momentum hu (+) at $t = 0.5$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

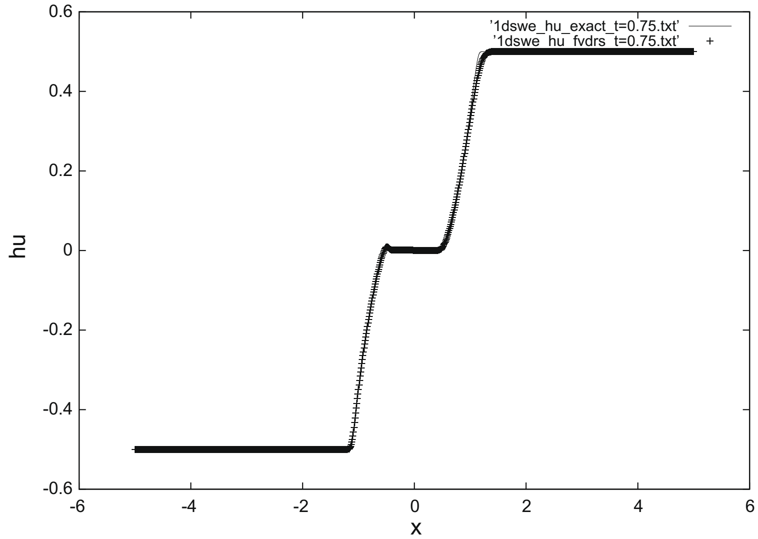


Fig. 20. Solution of momentum hu (+) at $t = 0.75$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

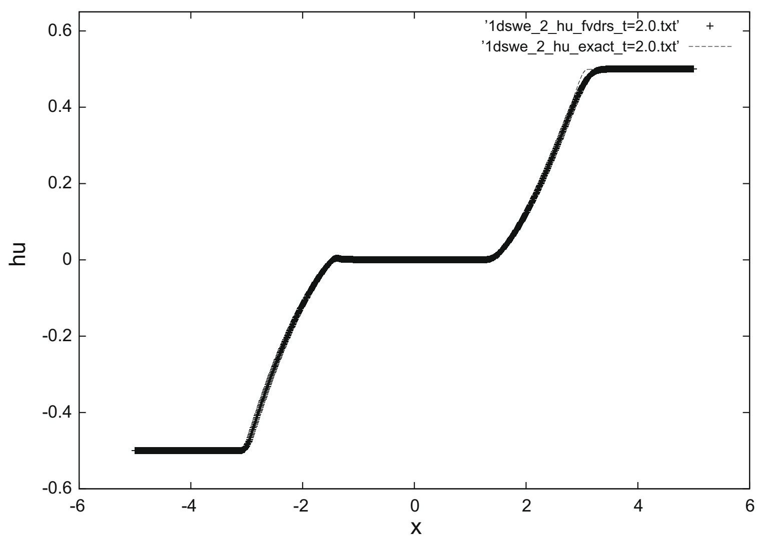


Fig. 21. Solution of momentum hu (+) at $t = 2$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

The spatial domain is $[0,1]$. The boundary conditions are outflow conditions using zeroth order extrapolation. Consider the bottom topography as

$$B(x) = 1.398 - 0.347\ \tanh(8x - 4) \tag{154}$$

The numerical solution to this problem is computed at $t = 0.25$ and is shown together with the initial condition in Figs. 23, 24 using FVDRS and is superimposed with the exact solution along with the bottom function $B(x)$.

7.5.4. Flow over an obstacle in one dimension

This test case is taken from [18]. In this example we study shallow water flow over an obstacle in one dimension. We use scaled equations so that $g = 1$. The spatial interval is $[-10.0, 10.0]$ with absorbing boundaries. The initial conditions are:

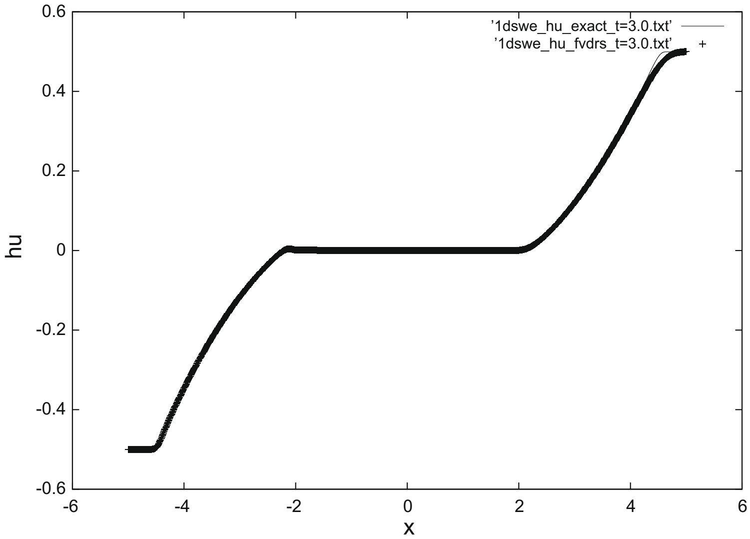


Fig. 22. Solution of momentum hu (+) at $t = 3$ using FVDRS with $\Delta x = 0.01$; solid line: exact solution.

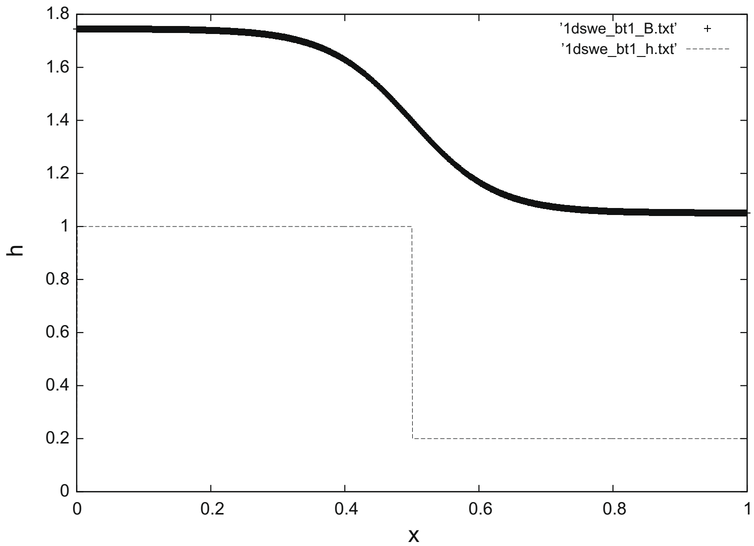


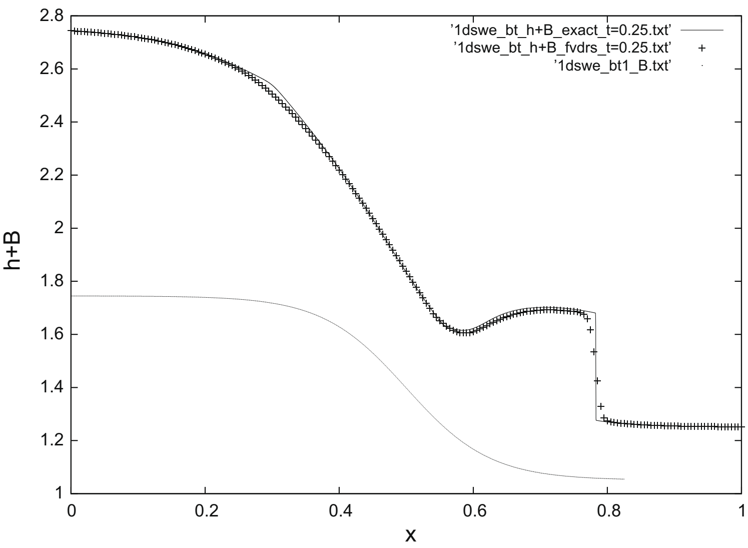
Fig. 23. Plot of h (dotted line) with bottom (+) at $t = 0$.

$$h(x,0) + B(x) = 1.0, \quad u(x,0) = 1.0 \tag{155}$$

and the bottom profile reads:

$$B(x) = 0.2 * \left(1 - \frac{x^2}{4}\right), \quad \text{for } -2 \leqslant x \leqslant 2; \; 0, \text{ otherwise} \tag{156}$$

The water depth is computed at $t = 5.0$ and $t = 10.0$ and is shown in Figs. 25–27 with $\Delta x = 0.01$ using FVDRS and is compared with the exact solution with bottom topography. This problem consists of a shock running upstream, another shock running downstream and a rarefaction wave running downstream [29].


'1dswe_bt_h+B_exact_t=0.25.txt'
'1dswe_bt_h+B_fvdrs_t=0.25.txt'
'1dswe_bt1_B.txt'
h+B
x
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0
0.2
0.4
0.6
0.8
1

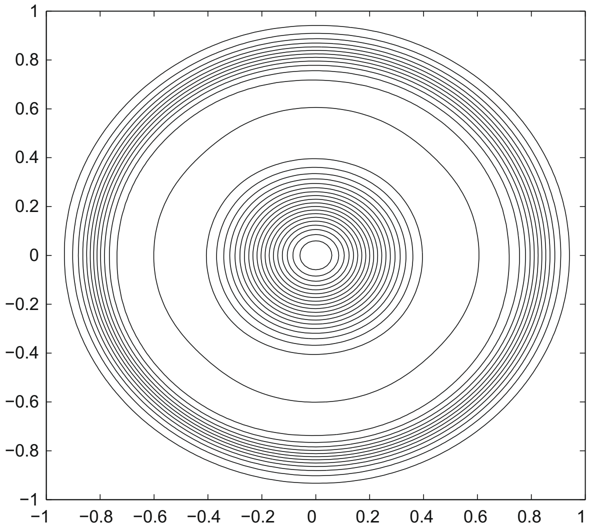


Fig. 28. Radial dam-break problem: solution at $t = 0.25$ with 50×50 grid.

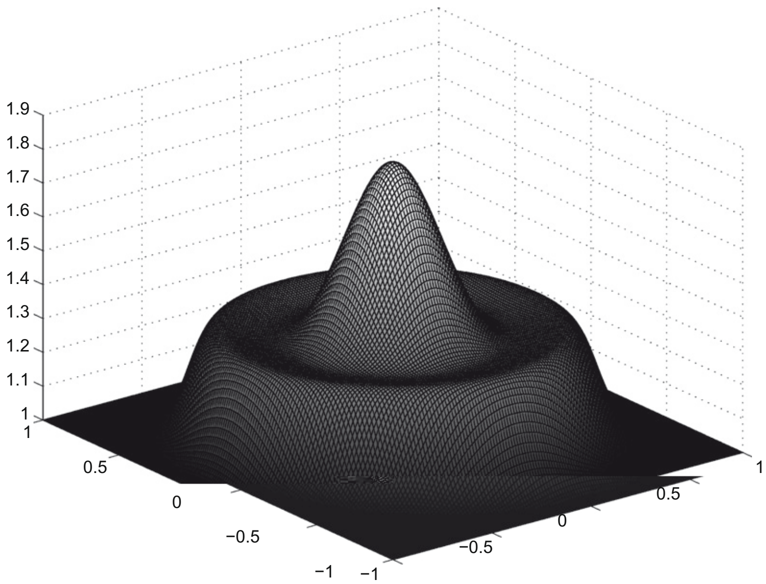


Fig. 29. Radial dam-break problem: solution at $t = 0.25$ with 50×50 grid; surface plot.

8. Conclusions

A new Finite Variable Difference Method, in the framework of a relaxation system which converts a nonlinear conservation law into a system of linear convection equations with nonlinear source terms, is proposed in which the spatial difference for discretizing the convection term is optimized so that the total deviation of the numerical solution from the exact solution of the convection–diffusion equation is minimized, under the condition that roots of the characteristic equation of the resulting difference equation are always nonnegative to ensure numerical stability. This scheme captures flow features very accurately. This feature is demonstrated by bench-mark problems for Burgers equation in 1-D and 2-D. This algorithm is extended to two dimensional scalar conservation laws by generalizing the 1-D discrete velocity Boltzmann equation, as in Aregba-Driollet and Natalini [2]. This new algorithm is also extended to hyperbolic vector conservation/balance laws in one and two dimensions and the bench-mark test problems for simulating shallow water flows demonstrate the efficiency of this scheme in capturing the flow features accurately. This approach based on the Finite Variable Difference Method cou-

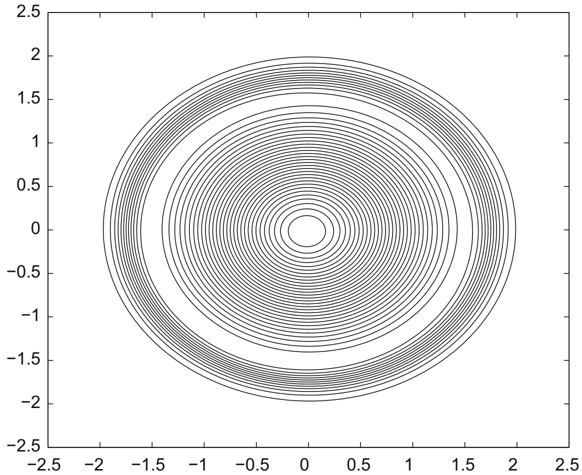
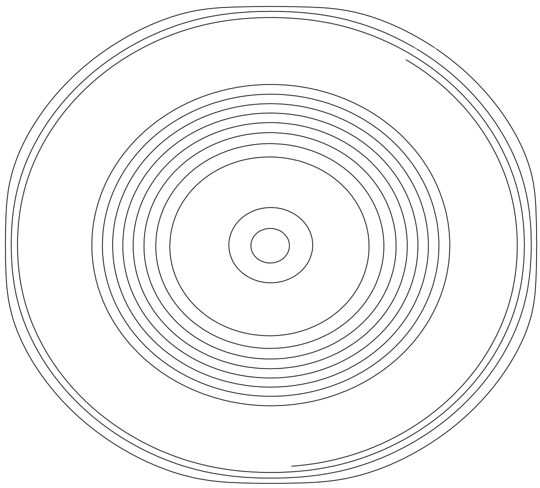


Fig. 30. Radial dam-break problem: solution at $t = 1.0$ with 125×125 grid.



pled with a Relaxation System provides an interesting alternative to the traditional numerical methods for solving hyperbolic-parabolic equations.

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